Split-Transformation Method to Derive Summation Invariants of 2D Objects under Projective Transformation Group

Nasereh Azhir and Jamaludin Md Ali

School of Mathematical Sciences
Universiti Sains Malaysia, 11800 Pulau Pinang, Malaysia

Abstract: This paper proposes a new method to derive summation invariant for 2D objects under projective groups of transformation. The Split-Transformation Method is defined to derive invariants under projective transformation with 8 degrees of freedom by splitting it into two transformations with equal or lower than 6 degrees of freedom. Their invariants are derived by giving potential variables for the standard actions of projective groups. This helps to solve the problem of nonlinearity of projective transformation group to derive invariants. Application of these invariants to discrete data, obtained from a sample of boundary of car contour, generates a pattern of similar classification under the projective transformation group.

Keywords: Group Action; Moving Frame; Projective Transformation; Summation Invariants.

1. Introduction

Geometric invariants have long been used in a wide variety of problems in computer vision and object recognitions [1-3]. Specifically, invariants under projective group of transformations have attracted more challenging research in this area. Given a curve associated to a manifold M may undergo projective transformation, can be described as action of PGL(3) lie group on the manifold M. Recently some effort has been devoted to derive invariants based on the moving frame method of Cartan. It has been formulated [4] as powerful algorithm for studying the geometric properties of sub-manifolds and their invariants under a transformation group. Although the existence of moving frame required freeness of the lie group, it has been solved by prolonging the moving frame action to jet spaces which lead to differential invariants [5].

However, the limitation of differential invariants [6] due to high sensitivity to noise lead to other types of invariants, such as integral invariants [7]. The authors [8] proposed the robust classified algorithm for planar curves under affine transformation which has been modified [9] as summation invariants. However, in the case of projective group, the problem of the existence of x and y terms in the dominator causes problems to derive invariants classically under the action of this group. In continuation of the integral potentials presented [10], given for special case of projection for the maximum 6 degrees of freedom, here, it is tried to derive new invariants of curves under projective transformation with 8 of freedom in a deductive approach. Splitting purposed transformation to transformations with lower degrees of freedom is defined which can derive their associated invariants more convenient.

The paper is structured as follows: the method of splitting the projective transformation group into two subgroups of $PGL(3)$ is given in section 1. Then this method is applied to derive summation invariants for planar objects under projective group of transformations with 8 degrees of freedom. An experimental result for the sample contour of the database is discussed in section 2 to evaluate of invariant by the proposed method. Finally, the conclusion of the work is given in section 3.

2. Split-transformation of group $PGL(3)$

Definition: [11]let G be a group that acts on a set X by

$$T_g(x) = T(g,x),\ \forall (g,x) \in G \times X$$

which satisfies the following axioms:

1. $T_e(x) = x$, $\forall x \in X$ and e the identity in G.

(DOI: dx.doi.org/14.9831/1444-8939.2015/3-4/MAGNT.32)

2. $T_{gh}(x) = T_g(T_h, x)$, $\forall g, h \in G$ and $x \in X$.

For each g in G, the bijection mapping $T_g : X \to X$ is defined by $T_g(x) = T(g, x)$, such that T_e is identity function and axiom 2 in definition means $T_g(T_h(x)) = T_{gh}(x)$, that is, each T_g is one-to-one and onto with the inverse $T_g^{-1} = T_{g^{-1}}$.

Thus, $\{T_g \mid g \in G\}$ is a group under function composition and function composition corresponds to groups composition in G.

Regarding the projective transformation group $\mathrm{PGL}(3)$, an attempt is made to split the transformation with 8 degrees of freedom into two of its subgroups with lower than 6 degrees of freedom.

Suppose the sampled points $\{(x_t, y_t) : n = 1, 2, ..., N\}$ is transformed by

$$\begin{pmatrix} \bar{x}_t \\ \bar{y}_t \\ \bar{z}_t \end{pmatrix}^T = C \begin{pmatrix} x_t \\ y_t \\ 1 \end{pmatrix} \quad (1)$$

Where

$$C = \begin{pmatrix} a & b & e \\ c & d & f \\ g & h & l \end{pmatrix} \quad (2)$$

belongs to $\mathrm{PGL}(3)$ denoted the projective transformation with 8 degrees of freedom, $\det C \neq 0$ and $(\frac{\bar{x}_t}{\bar{z}_t}, \frac{\bar{y}_t}{\bar{z}_t})$ represents the transformed point of (x_t, y_t) in Cartesian coordinates.

Action 1 induces almost everywhere the linear fractional action of $PGL(3)$ on R^2 in homogenous coordinates as the following map:

$$\varphi(x_t, y_t) = (\frac{ax_t + by_t + e}{gx_t + hy_t + l}, \frac{cx_t + dy_t + f}{gx_t + hy_t + l})$$

(3)

Proposition 2.1. Every projective transformation with 8 degrees of freedom can be split as the composition of two subgroups of the projective transformation with equal or less than 6 degrees of freedom.

Proof: Transformation $T_A : R^3 \to R^3$ is defined by $T_A(X) = A.X$ for A belongs to subgroup $G_1 \subset \mathrm{PGL}(3)$ such that

$$A = \begin{pmatrix} a & b & e \\ 0 & 1 & 0 \\ g_2 & h_2 & l_2 \end{pmatrix} \quad (4)$$

, $det(A) \neq 0$ for every $X \in R^3$. Transformation $T_B : R^3 \to R^3$ is defined by $T_B(x) = B.X$ which

$$B = \begin{pmatrix} 1 & 0 & 0 \\ c & d & f \\ g_1 & h_1 & l_1 \end{pmatrix} \quad (5)$$

in $G_2 \subset \mathrm{PGL}(3)$ and $det(B) \neq 0$. Both of transformations are projective transformations with 6 degrees of freedom. The composed transformation $T_{A \circ B} : R^3 \to R^3$ is the product of previous transformations T_A and T_B. As $T_{A \circ B}(X) = (T_A \circ T_B)(X) = (A.B).X$, so

$$A.B = C = \begin{pmatrix} a_2 & b_2 & e_2 \\ c & d & f \\ g_3 & h_3 & l_3 \end{pmatrix}, det(C) \neq 0 \quad (6)$$

and

$a_2 = a + bc + eg_1, b_2 = bd + eh_1, e_2 = bf + el_1,$

$g_3 = g_2 + dh_2 + g1l_2, h_3 = dh_2 + h1l_2, l_3 = fh_2 + l_1 l_2$

Finally, transformation T_C, the projective transformation with 8 degrees of freedom, is made by two transformations T_A and T_B.

In fact, projective transformations T_A and T_B on the curve $C(t) = (x(t), y(t))$ arise from two following actions of subgroups G_1 and G_2 in homogenous coordinates:

$$\phi_A(x_t, y_t) = (\frac{ax_t + by_t + e}{g_2 x_t + h_2 y_t + l_2}, \frac{y_t}{g_2 x_t + h_2 y_t + l_2}) \quad (7)$$

$$\phi_B(x_t, y_t) = (\frac{x_t}{g_1x_t + h_1y_t + l_1}, \frac{cx_t + dy_t + f}{g_1x_t + h_1y_t + l_1}) \quad (8)$$

These group's actions are prolonged to jet spaces by the following potential variables.

Definition2.2. The potential of order k (here k = 2) for N points of curve $C(t)$ under group transformation 4 is defined by:

$$p^{ij} = \sum_{t=1}^{N} \frac{x_t^i}{y_t^j}, i \leq j \leq k. \quad (9)$$

Definition2.3. The Potential of order k associated to points of curve under group transformation 5 is defined by:

$$q_{ij} = \sum_{t=1}^{N} \frac{y_t^i}{x_t^j}, i \leq j \leq k. \quad (10)$$

The actions ϕ_A and ϕ_B are prolonged on the points set of the curve in R^2 to potential variables 8 and 9.

The corresponding jet space J_1^2 for transformation group ϕ_A with potentials of order 2 is coordinated by:

$$J_1^2 = (x_1, y_1, x_t, y_t, p^{01}, p^{11}, p^{02}, p^{12}) \quad (11)$$

The normalized equations for prolonged jet space J_1^2 is supposed by

$$(\overline{x}_1, \overline{x}_N, \overline{y}_1, \overline{y}_N, \overline{p}^{01}, \overline{p}^{11}) = (0,1,1,1,0,0) \quad (12)$$

Solving the normalized equations 12 for the transformed variables leads to the moving frame $\{a,b,e,g,h,l\}$ for action ϕ_A or transformation T_A. Substituting the moving frame $\{a,b,e,g,h,l\}$ in $\overline{p^{22}}$ derives the summation invariant I_{22}^1 for group transformation T_A as follows:

$$I_{22}^1 = y_N^2(p^{22}p^{01^2}y_{t_0}^2 - 2p^{22}p^{01}y_{t_0}N +$$
$$p^{22}N^2 - Nx_{t_0}^2 p^{01^2} + 2Nx_{t_0}p^{01}p^{11}$$
$$-Np^{11^2} + p^{02}x_{t_0}^2N^2 - 2p^{02}x_{t_0}Np^{11}y_{t_0}$$
$$+p^{02}p^{11^2}y_{t_0}^2 + 2p^{12}p^{01}y_{t_0}x_{t_0}N$$
$$-2p^{12}p^{01}y_{t_0}^2p^{11} - 2p^{12}x_{t_0}N^2$$
$$+2p^{12}p^{11}y_{t_0}N)/(-x_{t_0}p^{01}y_N + x_{t_0}N$$
$$+p^{01}y_{t_0}x_N - p^{11}y_{t_0} - x_NN + p^{11}y_N)^2 \quad (13)$$

On the other hand, similarity, the corresponding potential jet space J_2^2 for ϕ_B is coordinated by:

$$J_2^2 = (x_1, y_1, x_t, y_t, q^{01}, q^{11}, q^{02}, q^{12}) \quad (14)$$

The following normalized equations are solved for transformed variables in order to derive the moving frame $\{c,d,f,g,h,l\}$ for action ϕ_B or transformation T_B.

$$(\overline{x}_1, \overline{x}_N, \overline{y}_1, \overline{y}_N, \overline{q}^{01}, \overline{q}^{11}) = (1,1,0,1,0,0) \quad (15)$$

Substituting the moving frame $\{c,d,f,g,h,l\}$ in q_{22} gives the summation invariant I_{22}^2 for transformation T_B as the following formulae:

$$I_{22}^2 = x_N^2(-Nq_{11}^2 + 2Nq_{11}q_{01}y_{t_0} -$$
$$Nq_{01}^2y_{t_0}^2 + q_{22}q_{01}^2x_{t_0}^2 - 2q_{22}q_{01}x_{t_0}N$$
$$+q_{22}N^2 + q_{02}q_{11}^2x_{t_0}^2 - 2q_{02}q_{11}x_{t_0}y_{t_0}N$$
$$+q_{02}y_{t_0}^2N^2 - 2q_{12}q_{01}x_{t_0}^2q_{11}$$
$$+2q_{12}q_{01}x_{t_0}y_{t_0}N + 2q_{12}Nq_{11}x_{t_0}$$
$$-2q_{12}y_{t_0}N^2)/(-q_{11}x_N + q_{01}x_Ny_{t_0}$$
$$-y_Nq_{01}x_{t_0} + y_NN + q_{11}x_{t_0} - y_{t_0}N)^2 \quad (16)$$

It looks not to be easy deriving invariant for a given curve c, which is transformed to curve c' under projection 2 with 8 degrees of freedom, finding such a function f that satisfies in $f(X) = f(gX)$ for each $g \in \mathrm{PGL}(3)$ and $X \in R^2$ based on proposed potentials 9 and 10.

To solve this problem, a deductive approach is splitting this transformation based on proposition 2 making an image like c_1 under transformation ϕ_B which can be transformed to c' (under transformations with lower than 6 degrees of freedom). Regarding to following lemma, it is convenient derive invariant f under this transformation.

Lemma 2.4: Suppose the action $T: G \times X \to X$ can be split into two actions $T_1 : G_1 \times X \to X$ and $T_2 : G_2 \times X \to X$, for two subgroups G_1 and G_2 of PGL(3).

Then for every $x \in X$ and $g_1, g_2 \in G_2$,

1. If $T = T_1 \circ T_2$ then every invariant function $f_1 : X \to R$ under action T_1 satisfies in $f_1(T(g_1g_2x)) = f_1(T_2(g_2x))$.
2. If $T = T_2 \circ T_1$ then every invariant function $f_2 : X \to R$ under action T_2 satisfies in $f_2(T(g_2g_1x)) = f_2(T_1(g_1x))$.

Proof: As f_1 is invariant under action T_1 then $f_1(x(t), y(t)) = f_1(g_1(x(t), y(t))$ for every $g_1 \in G_1$ and $(x, y) \in X$. Also group G is split of subgroups G_1 and G_2, so especially for the image of curve under action T_2 (under composite actions T_1 and T_2), the invariant f_1 gives

$$f_1(T_{g_1 \circ g_2}(x(t), y(t)) = f_1(g_1g_2(x, y)) = f_1(g_2(x(t), y(t)).$$

Case 2 is proved in a similar way as 1.

According to proposition, as I_{22}^1 is invariant under action ϕ_A, so $I_{22}^1(x(t), y(t)) = I_{22}^1(\phi_A(x(t), y(t)).$

Also as f_2 is invariant under transformation ϕ_B then $f_2(x(t), y(t)) = f_2(\phi_B(x(t), y(t))$ for every (x, y) in X, especially for image of curve under transformation ϕ_A. So f_2 will be invariant under composition of functions $f_2(\phi_{A \circ B}(x(t), y(t)) = f_2(\phi_B(x(t), y(t))$ which leads $f_2(\phi_C(x(t), y(t)) = f_2(\phi_B(x(t), y(t)).$

Assume that curve $c = (x_t, y_t)$ is transformed to $c' = (\bar{x}_t, \bar{y}_t)$ under projective transformation (T) with 8 degrees of freedom. To evaluate the proposed invariants, first c is transformed into an image based on split-transformation method, then derived invariants can be used for c' and this image.

Therefore, this image is made in the two following cases:

- **Case 1**

To use the invariant I_{22}^1, the transformation matrix C is defined as

$$\begin{pmatrix} a+bc & bd & bf+e \\ c & d & f \\ g+ch & hd & fh+l \end{pmatrix}. \quad (17)$$

which is the product of matrices

$$A = \begin{pmatrix} a & b & e \\ 0 & 1 & 0 \\ g & h & l \end{pmatrix} \quad (18)$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}. \quad (19)$$

Solving equations obtained from setting equal matrix C and transformation T, the values of matrices A and B will obtain. According to proposition, the values of I_{22}^1 will not change for c' and image of curve under transformation A.

In fact, purposed image is transformation of c under group action matrix A.

- **Case 2**

To evaluate the invariant I_{22}^2, the transformation matrix C is defined by

$$\begin{pmatrix} a & b & e \\ ca+fg & cb+d+fh & ce+fl \\ g & h & l \end{pmatrix}. \quad (20)$$

This is product of matrices B and A defined in 18 and 19.

To use the method, the invariant I_{22}^1 is evaluated for images of curves under transformations B and C. According to the proposition, their values will be equal.

(DOI: dx.doi.org/14.9831/1444-8939.2015/3-4/MAGNT.32)

Example: Suppose curve $c(t) = (t^2 + 1, 2t + 5)$, as it is shown in figure 1.a, is transformed to $c'(t) = (\frac{5t^2 - 4t - 2}{3t^2 - 2t}, \frac{13t^2 12t - 9}{3t^2 - 2t})$ by action $T(x, y) = (\frac{x + 2y + 1}{x + y + 1}, \frac{2x - y + 1}{x + y + 1})$.

Assuming that the transformation matrix is 17, the matrices A and B are obtained so that the group actions T_1 and T_2 related to these matrices are given as follows:

$$T_1(x, y) = (\frac{5x - 2y + 3}{3x - y + 2}, \frac{y}{3x - y + 2}) \quad (21)$$

and

$$T_2(x, y) = (x, 2x - y + 1) \qquad (22)$$

The transformed image of c' under action T_2 is shown in Fig. 1 denoted as c_1. Evaluating the invariant I_{22}^1 for c' and c_1 gives the same numerical values as 121.8518866.

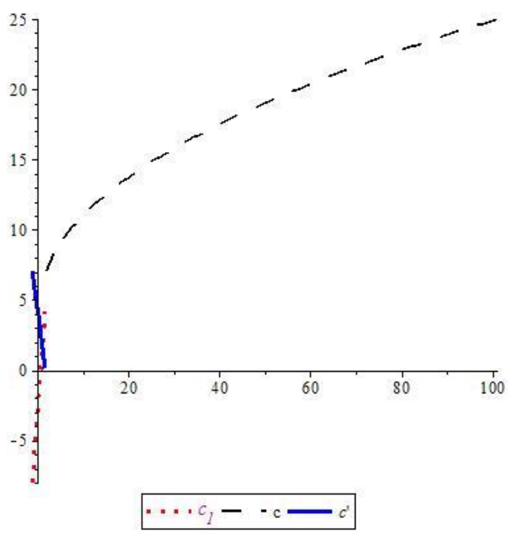

Fig. 1. Split curve c(t) under transformations A and B

3. Experimental results

A sample of car contour is chosen from data base with N=334 number of points as shown in figure 2. It is projected under transformation group

$$C = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad (23)$$

(figure 3) which is split as the product of two projective transformations, according to 21 and 22 denoted as c' and c_1.

The value of the invariant I_{22}^1 is calculated for the car shape c' and its image under transformation A. As illustrated in the table 1, they have the same values.

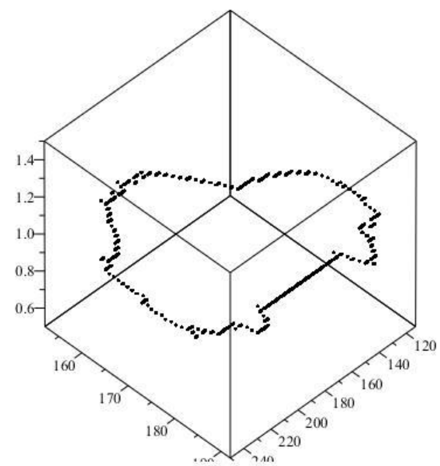

Fig. 2. Error! No text of specified style in document.**-a:** Sample of car contour from data base with N=334 points

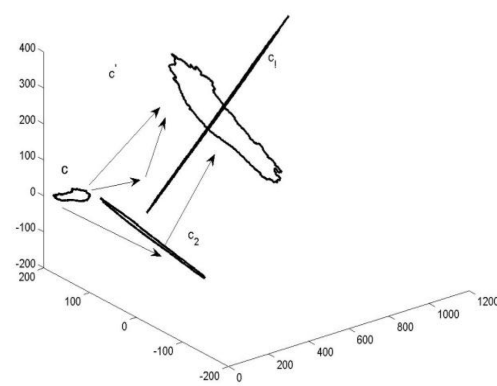

Fig. 3: Car contour under two different split-transformations

(DOI: dx.doi.org/14.9831/1444-8939.2015/3-4/MAGNT.32)

In addition, the estimate of I_{22}^{1} is calculated for noisy data points of contours under two levels of white Gaussian noise $\sigma = 0.1$ and $\sigma = 0.5$. (Figures 4 and 5). Their results are equal.

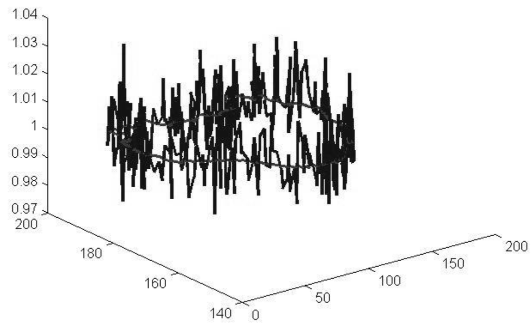

Fig. 4: Car under white Gaussian noise $\sigma = 0.1$

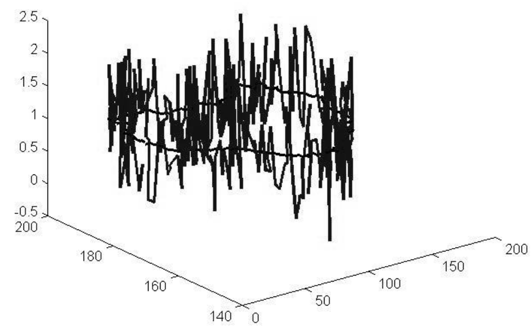

Fig. 5: Car under white Gaussian noise $\sigma = 0.5$.

In addition, the split transformation method is applied for curve under transformation 20, which is related to two transformations

$$T_1(x,y) = \left(\frac{x+2y+3}{2x+y+1}, \frac{y}{2x+y+1}\right),$$

and (21)

$$T_2(x,y) = \left(x, \frac{1}{5}(x-9y+2)\right)$$

Evaluating the invariant I_{22}^{2} for curves c' and c_2 gives the equal results as shown in table 2.

4. Conclusion

In this paper, a deductive method is proposed for deriving summation invariants of planar objects subjected to projective transformation group with 8 degrees of freedom. By splitting the group of projective transformation into two subgroups of the projective transformation with lower 6 degrees of freedom, the problem of deriving invariants for the action of projective groups on R^2 can be solved. Derived invariants by this method have been applied for car contour sample, which can be used in the classification of objects under projective transformation groups and in solving problems of object recognitions in computer vision. The addition of noise to the data set of contour shapes has shown their resistance of invariants to noise under projective transformations.

Table 1: Invariant I_{22}^{1} for car contour c under two projective transformations with split-transformation method

N=334	σ	c_1	c'
I_{22}^{1}	0	2.580117482	2.5801174
I_{22}^{1}	0.1	4.117281010	4.210^{7}
I_{22}^{1}	0.5	1.334691251	1.3710^{8}

(DOI: dx.doi.org/14.9831/1444-8939.2015/3-4/MAGNT.32)

Table 2: Invariant I_{22}^{2} for contour c under two projections 24 under Noise levels $\sigma = 0.5.$, 0.1 with split-transformation method

N=334	σ	c_1	$c^{'}$
I_{22}^{2}	**0**	129.0302860	129.0302860
I_{22}^{2}	**0.5**	933.3626201	933.3626201

References

1. Mundy J.L. and A. Zisserman,1992, Geometric invariance in computer vision. MIT press Cambridge, MA.Vol. 92.
2. Weiss I,1993, Geometric invariants and object recognition. International journal of computer vision, 1993. 10(3):207-231.
3. Tannenbaum A, Sapiro G and Olver, P. J, 1997. Invariant geometric evolutions of surfaces and volumetric smoothing. SIAM Journal on Applied Mathematics, 57(1):176-194.
4. Fels, M and Olver P.J,1999, Moving coframes: II. Regularization and theoretical foundations. Acta Applicandae Mathematica,55(2):127-208.
5. Olver P.J,2001, Moving frames-in geometry, algebra, computer vision, and numerical analysis. LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES, 2001: 267-298.
6. Calabi E,Olver P.J, Shakiban Ch, Tannenbaum A and Haker S,1998,Differential and numerically invariant signature curves applied to object recognition,International Journal of Computer Vision,Springer,26,2,107--135.
7. Hann C.a and Hickman M,1998, Projective curvature and integral invariants. Acta Applicandae Mathematica, 2002. 74(2):177-193.
8. Feng S, Kogan I and Krim H,2010, Classification of curves in 2D and 3D via affine integral signatures. Acta applicandae mathematicae, 2010. 109(3):903-937.
9. Lin W-Y, Boston N and Hu Y.H,2005, Summation invariant and its applications to shape recognition. in Acoustics, Speech, and Signal Processing, 2005.
10. Azhir, N., & Ali, J. M. (2014, July). Integral invariants for curves in 2D with respect to projective groups. In *PROCEEDINGS OF THE 21ST NATIONAL SYMPOSIUM ON MATHEMATICAL SCIENCES (SKSM21): Germination of Mathematical Sciences Education and Research towards Global Sustainability*(Vol. 1605, pp. 524-528). AIP Publishing.
11. Olver P.J, 1995, Equivalence, invariance and symmetry,Cambridge Univ. Press

(DOI: dx.doi.org/14.9831/1444-8939.2015/3-4/MAGNT.32)