Bayesian Inference for the Scale Parameter of the Discrete Rayleigh Distribution

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Abstract: In this paper, Bayesian inference is devoted for the scale parameter of the discrete Rayleigh distribution. Based on a beta prior, squared error loss function (SELF) and general entropy loss function (GELF), and Bayesian estimators are obtained in bounded and explicit forms. Following the modified posterior, high posterior density (HPD) credible intervals, testing for both simple and composite hypothesis are involved. As a complement to the Bayesian inference, prediction for the future ordered observation is also included. The theoretical frame work is illustrated through the numerical results found out applying real lifetime's data and an extensive simulation study.

Keywords: Discrete Rayleigh distribution; probability mass function; Bayesian estimation; loss functions; high posterior credible interval.

1. Introduction

In reliability and survival analysis modeling, it is common to treat failure data as being continuous. However, in many other practical situations, it is impossible or inconvenient to measure the lifetimes on a continuous scale. Hence, discrete time random variables find their way into reliability and survival analysis. Examples are the number of runs, cycles, or shocks of a device or a system before it fails. Even for a continuous operation, failure data may be only obtained at periodic time points. This gives rise to appropriate discrete lifetime's models.

Several discrete lifetimes distributions are mathematically constructed by discretizing their conjugate continuous models using various methods as in Lai[1].Some of the important discrete life time's models are employed by many authors; examples are Nakagawa and Osaki [2], Stein and Dattero [3], Roy [4], Krishna and Punder [5] and Aghbabai et al. [6].

The discrete Rayleigh distribution is derived from the continuous Rayleigh distribution by Roy [7].It is defined on the nonnegative integer valued random numbers by the probability mass function (pmf).

$$P(X=x)=\begin{cases}\theta^{(x+1)^2}-\theta^{(x)^2}, & x=0,1,2,\dots\\ 0 & , otherwise\end{cases} \tag{1}$$

Where, the scale parameter $\theta: 0<\theta<1$

The corresponding reliability and hazard functions are:

$$R(X=x)=\theta^{x^2}, x=0,1,2,\dots \tag{2}$$

$$h(X=x)=-lin\left(\theta^{x^2}\right)\ , x=0,1,2,\dots \tag{3}$$

This model can be applied in reliability determination of sold shaft, and as a distribution of stress and strength data in reliability analysis, it is also considered a lifetime model for complex systems. Details about these applications are implemented in Roy and Gupta [8] and Roy and Dasgupta [9].

Setting: $\theta=\exp(-\lambda)\,,\lambda>0$, implies that the reliability function defined in (2) describes the reliability function of a continuous Rayleigh distribution given by

$$R(X=x)=\exp(-\lambda x^2) \tag{4}$$

The scale parameter, θ of the discrete Rayleigh distribution can be estimated using ordinary classical maximum likelihood, moment and empirical methods as considered by Roy [7].

Let $X_1,X_2,\dots,X_n$ be n random sample observations from the discrete Rayleigh distribution with (pmf) given in (1).Then the likelihood function of θ is given by

$$L(X|\theta)=\theta^{\sum_{i=1}^{n}{X_i}^2}\prod_{i=1}^{n}(1-\theta^{2X_i+1}) \tag{5}$$

Hence, the maximum likelihood estimator (MLE) θ_{ml} of θ is obtained as the solution of the equation:

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$$\sum_{i=1}^{n}(X_i+1)^2=\sum_{i=1}^{n}\frac{(2X_i+1)}{(1-\theta^{2X_i+1})} \quad (6)$$

Numerical solution is needed to find θ_{ml}.

Under the method of moments, the sample mean, $\bar{X}$ will be equated with the population mean:

$$\mu(\theta)=\theta+\theta^4+\theta^9+\cdots \quad (7)$$

Solving for θ from $\bar{X}=\mu(\theta)$, we get θ_{mom}, the moment estimator of θ.

As an integral part of the classical statistical inference, Bayesian approach has been currently used in many studies of reliability and survival analysis. Examples are included in Jaheen [10], Soliman et al. [11], Soliman and Al-Aboud [12], and Gouet et al. [13].

A distinctive feature of the Bayesian analysis is the presence of the amount of prior information about the parameter of interest. In order to select a single value δ as the Bayesian estimator of θ, a loss function must be specified. A frequently used loss function is the squared error loss function (SELF) defined by

$$L_s(\theta,\delta)=(\theta-\delta)^2 \quad (8)$$

Hence, Berger [14] has obtained the Bayesian estimator as the posterior mean.

In many practical situations, it appears more realistic to express the loss function in terms of the ratio $\frac{\theta}{\delta}$. In this case, a useful symmetric loss function is the general entropy loss function (GELF) defined by Callabri and Pulcini [15] by

$$L_G(\theta,\delta)=(\frac{\delta}{\theta})^{M}-\theta\,\mathrm{Log}\left(\frac{\delta}{\theta}\right)-1,\ M>0 \quad (9)$$

Whose minimum occurs at $\theta=\delta$. This loss function is a generalization of the entropy loss function and used by many authors as Soliman and Elkahlout [16], and Migdadi and Al-Batah [17].

Hence, the Bayesian estimator under the GELF is given by

$$\delta=E_{\Pi}(\theta^{-M})^{\frac{-1}{M}} \quad (10)$$

2. Prior and Posterior Functions

In this section, assuming the random sample $X_1,X_2,\ldots,X_n$ observations are from the discrete Rayleigh, a proposed prior distribution and a corresponding posterior function are derived for the scale parameter θ.

Based on the likelihood function obtained in (5), a reasonable prior of θ is proposed to be the $Beta(a,b)$ distribution given by

$$\Pi(\theta)=\frac{1}{beta\,(a,b)}\theta^{a-1}(1-\theta)^{b-1},\ 0<\theta<1,\quad a,b\in R \quad (11)$$

Where: $beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

If $a=0,b=1$, then we have the ordinary uniform non informative prior

$$\Pi(\theta)=\frac{1}{\theta},\theta>0 \quad (12)$$

The mean and the variance of the prior given in are

$$E(\theta)=\frac{a}{a+b} \quad (13)$$

$$V(\theta)=\frac{ab}{(a+b+1)(a+b)^2} \quad (14)$$

The mod for this prior is given by

$$\Pi_{mod}=\frac{a-1}{a+b-2} \quad (15)$$

Combining the likelihood function in (5), and the prior function in (11), the posterior function of θ given the random sample $\boldsymbol{X}$ is given by

$$\Pi(\theta|\boldsymbol{X})=\frac{\theta^{\sum_{i=1}^{n}X_i^2+a-1}(1-\theta)^{b-1}\prod_{i=1}^{n}(1-\theta^{2X_i+1})}{\int_0^1\theta^{\sum_{i=1}^{n}X_i^2+a-1}(1-\theta)^{b-1}\prod_{i=1}^{n}(1-\theta^{2X_i+1})d\theta} \quad (16)$$

Using the transformations:

$$\theta=(R(X))^{\frac{1}{X^2}},\qquad R(X)>0 \quad (17)$$

$$\theta=(e^{-h(X)})^{\frac{1}{X^2}},\qquad h(X)>0 \quad (18)$$

The posterior functions for both the reliability and the hazard functions are readily derived.

3. Bayesian Point Estimation

In this section, Bayesian estimators for the scale parameter θ are obtained and modified under both, the squared and the general entropy error loss functions.

Under SELF, the Bayesian estimator θ_S, for θ is given by the posterior mean by

$$\boldsymbol{\theta_S=\frac{\int_0^1\theta^{\sum_{i=1}^{n}X_i^2+a}(1-\theta)^{b-1}\prod_{i=1}^{n}(1-\theta^{2X_i+1})\ d\theta}{\int_0^1\theta^{\sum_{i=1}^{n}X_i^2+a-1}(1-\theta)^{b-1}\prod_{i=1}^{n}(1-\theta^{2X_i+1})d\theta}} \quad (19)$$

And under (GELF), the Bayesian estimator θ_G for θ is given by

$$\boldsymbol{\theta_G=\left(\frac{\int_0^1\theta^{\sum_{i=1}^{n}X_i^2+a-1-M}(1-\theta)^{b-1}\prod_{i=1}^{n}(1-\theta^{2X_i+1})\ d\theta}{\int_0^1\theta^{\sum_{i=1}^{n}X_i^2+a-1}(1-\theta)^{b-1}\prod_{i=1}^{n}(1-\theta^{2X_i+1})d\theta}\right)^{\frac{-1}{M}}} \quad (20)$$

The estimators in (15) and (16) cannot be in an explicit closed form and numerical methods are needed to obtain the integrals in both numerators and denominators of (15) and (16). Therefore, we proceed to indicate upper and lower bounds for $\theta_{S,}\theta_G$ mathematically using the following theorems:

Theorem (1): The Bayesian estimator θ_S for θ is bounded with lower bound given by

$$\theta_{SL} = \frac{\sum_{i=1}^{n}(-1^{i})\binom{n}{i}\text{beta}\,(\sum_{j=1}^{n}X_j^2+a+1+im_1,b)}{\sum_{i=1}^{n}(-1^{i})\binom{n}{i}\text{beta}\,(\sum_{j=1}^{n}X_j^2+a+im_2,b)} \qquad (21)$$

and an upper bound given by

$$\theta_{SU} = \frac{\sum_{i=1}^{n}(-1^{i})\binom{n}{i}beta\,(\sum_{j=1}^{n}X_j^2+a+1+im_2,b)}{\sum_{i=1}^{n}(-1^{i})\binom{n}{i}beta\,(\sum_{j=1}^{n}X_j^2+a+im_1,b)} \qquad (22)$$

Theorem (2): The Bayesian estimator θ_G for θ is bounded with lower bound given by

$$\boldsymbol{\theta_{GL}} = \left(\frac{\sum_{i=1}^{n}(-1^{i})\binom{n}{i}beta(\sum_{j=1}^{n}X_j^2+a-M+im_1,b))}{\sum_{i=1}^{n}(-1^{i})\binom{n}{i}beta(\sum_{j=1}^{n}X_j^2+a-M+im_2,b)}\right)^{\frac{-1}{M}} \qquad (23)$$

and an upper bound given by

$$\boldsymbol{\theta_{GU}} = \left(\frac{\sum_{i=1}^{n}(-1^{i})\binom{n}{i}beta(\sum_{j=1}^{n}X_j^2+a+1+im_2,b)}{\sum_{i=1}^{n}(-1^{i})\binom{n}{i}beta(\sum_{j=1}^{n}X_j^2+a+1+im_1,b)}\right)^{\frac{-1}{M}} \qquad (24)$$

Proofs of theorems (1) and (2) are given in the Appendix (1).

Since $(1-\theta^{2X_i+1}) = (1-\theta)f_i(\theta)$ where $f_i(\theta)$ is a polynomial of degree $2X_i$ of θ. This implies,

$$\prod_{i=1}^{n}(1-\theta^{2X_i+1}) = (1-\theta)^n\prod_{i=1}^{n}f_i(\theta)$$

As the sample size n increases $\prod_{i=1}^{n}f_i(\theta)$ converges to 1, implies the posterior function defined in (11) becomes

$$\Pi(\theta|\boldsymbol{X}) = \frac{\theta^{\sum_{i=1}^{n}{X_i}^2+a-1}(1-\theta)^{n+b-1}}{\int_0^1\theta^{\sum_{i=1}^{n}{X_i}^2+a-1}(1-\theta)^{n+b-1}d\theta} \qquad (25)$$

Which is also represents a beta $(\sum_{i=1}^{n}{X_i}^2 + a, n+b)$ distribution, and can be considered as a conjugate prior for θ.

Using (25), Bayesian point estimators for θ under SELF and GELF are respectively given by

$$\theta_S = \frac{\sum_{j=1}^{n}X_j^2+a}{\sum_{j=1}^{n}X_j^2+a+n+b} \qquad (26)$$

$$\theta_G = \left(\frac{Beta\,(\sum_{j=1}^{n}X_j^2+a+b-M \quad ,n+b\ )}{Beta\,(\sum_{j=1}^{n}X_j^2+a \quad ,n+b\ )}\right)^{\frac{-1}{M}} \qquad (27)$$

4. HPD Credible Intervals

In this section, HPD credible interval for the parameter θ is derived.

A set $C\subset(0,1)$ of the form $C=\{\theta:\Pi(\theta|X)\geq c_\alpha\}$, where c_α is the largest constant such that $P(\theta\in C)=1-\alpha$, is called a $100\%(1-\alpha)$ HPD credible interval for θ. Due to the unimodality of the posterior function defined in (25).The $100\%(1-\alpha)$ HPD credible interval $[c_1, c_2]$ for θ must simultaneously satisfy:

$\Pi(c_1|X)-\Pi(c_2|X)=1-\alpha$ and $\Pi(c_1|X)=\Pi(c_2|X)$. This leads that c_1, c_2 must satisfy the following equations:

$$\int_{c_1}^{c_2}\Pi(\theta|X)\,d\theta = 1-\alpha \qquad (28)$$

$$c_1^{\sum_{j=1}^{n}X_j^2+a-1}(1-c_1)^{b-1} = c_2^{\sum_{j=1}^{n}X_j^2+a-1}(1-c_2)^{b-1} \qquad (29)$$

Substituting for $\Pi(\theta|X)$ in (28) using (25) and simplifying, implies c_1, c_2 satisfy the following two nonlinear equations

$$\frac{1}{beta\,(a,b)}\int_{c_1}^{c_2}\theta^{\sum_{j=1}^{n}X_j^2+a-1}(1-\theta)^{n+b-1}\,d\theta = 1-\alpha \qquad (30)$$

$$\left(\frac{1-c_1}{1-c_2}\right)^{n+b-1} = \left(\frac{c_2}{c_1}\right)^{\sum_{j=1}^{n}X_j^2+a-1} \qquad (31)$$

The integral in equation (30) can be simplified using the incomplete beta function. For details, see appendix (2).

5. Prediction

In this section, prediction for the future ordered observation is obtained.

Let y_1 , $y_2, \dots$, y_m be an ordered sample of m future observations independently drawn from the discrete Rayleigh distribution with pmf given in (1). The pmf of the kth $(1\leq k\leq m)$ order statistic, y_k is

$P_k(y|\theta) = c\left(1-\theta^{y^2}\right)^{k-1}\left(\theta^{y^2} -\right.$ *θy+12(θy2) m−k ,y=0,1,2,....* (31)

where: $c = \frac{m!}{(k-1)!(m-k)}$

Using the binomial expansion : $(1-$ *θy2k−1=i=1k−1k−1i(−1)iθiy2* and simplifying of (31), implies the (pmf) of kth order statistics y_k becomes

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$P_k(y|\theta) =$
$c(\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i\,(\theta^{(i+m-k+1)y^2} -$
$\theta^{(i+m-k+1)y^2+2y+1})$ (32)

Using (25), given the data X, the predictive pmf of y_k can be expressed as

$P_k(y|X) =$
$\frac{c}{beta\,(w\ ,b)}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i\int_0^1(\theta^{(i+m-k+1)y^2+w} -$
$\theta(i+m-k+1)y2+2y+1+w)(1-\theta)b-1$
$d\theta$(33) where $w = \sum_{j=1}^{n} X_j^2 + a - 1$

Simplifying the integrals in (33), we have

$$P_k(y|X) = c1\sum_{i=0}^{k-1}(-1^i)\binom{k-1}{i}(beta((i+m-k+1)y^2+w+1,b) - beta((i+m-k+1)y+2y+w+2,b)$$

Where $c1 = \frac{c}{beta\,(w\ ,b)}$

And hence, the expected value of y_k can be obtained as

$E(y_k = y) = \sum_{y=1}^{m} yP_k(y|X)$ (34)

6. Testing

In the Bayesian frame analysis. A decision between the null hypothesis $H_0: \theta \in U_0$, and the alternative $H_1: \theta \in U_1$ is based on the posterior probabilities corresponding to both U_0 and U_1. Let$P_0(X) = P_\Pi(\theta \in U_0|X)$,
and $P_1(X) = P_\Pi(\theta \in U_1|X)$.
Then a decision is in favor of H_1 if and only if $P_0(X) < P_1(X)$. Using the posterior function in(25),H_0 is rejected if

$\int_{\theta\in U_0}\theta^{\sum_{i=1}^{n}X_i{}^2+a-1}(1-\theta)^{n+b-1}\,d\theta <$
$\int_{\theta\in U_1}\theta^{\sum_{i=1}^{n}X_i{}^2+a-1}(1-\theta)^{n+b-1}\,d\theta$ (35)

when, the "$0 - k_i$" loss function defined by

$l(\theta,a_i) = \begin{Bmatrix} 0, \theta \in U_i \\ k_i, \theta \in U_{1-i}\end{Bmatrix}, k_i > 0, i = 0{,}1$ (36)

where a_i represents the action for accepting $H_i, i = 0{,}1$,the posterior expected losses of a_0 , a_1 are $k_0P_1(X)$ and $k_1P_0(X)$ respectively. Hence, the Bayesian decision is that corresponding to the smallest expected loss.

If $U_0 \cup U_1 = (0{,}1)$. Then following Fernandez [18], the action a_1 is taken and H_0 is rejected if

$\int_{\theta\in U_1}\theta^{\sum_{i=1}^{n}X_i{}^2+a-1}(1-\theta)^{n+b-1}\,d\theta >$
$\frac{k_1}{k_0+k_1}$ (37)

when both hypotheses are simple: $H_0: \theta = \theta_0, H_1: \theta = \theta_1$,then H_0 is rejected if

$\frac{L(X|\theta_1)}{L(X|\theta_0)} > \frac{k_1q_0}{k_0q_1}$ (38)

where $L(X|\theta_0), L(X|\theta_1)$ are the likelihood functions given θ_0, θ_1 respectively, and q_0, q_1 are the prior probabilities given θ_0, θ_1 respectively. Hence, using (5) and (11),H_0 is rejected if

$\left(\frac{\theta_1}{\theta_0}\right)^{\sum_{j=1}^{n}X_j^2}\frac{\prod_{i=1}^{n}(1-\theta_1{}^{2X_i+1})}{\prod_{i=1}^{n}(1-\theta_0{}^{2X_i+1})} >$
$\frac{k_1}{k_0}\left(\frac{\theta_0}{\theta_1}\right)^{a-1}(\frac{1-\theta_0}{1-\theta_1})^{b-1}$ (39)

7. Numerical Results

In this section, the developed theoretical findings are illustrated through applying real lifetime's data and according to a simulation study.

7.1 Real life time's data

The following data are real lifetimes given as the number of cycles of use until a testing electrical machine failed, introduced by Nelson [19] and re analyzed by Lawless [20] at the failure code 9.

2223,4329,3112,7846,3504,2568,2471,3214,3034,3034,6976,2400,1167,1925,1990,2551,3478.

The above data are scaled dividing each by 1000,and then taken the least integer numbers of the outcomes to have the data:2,4,3,8,4,3,2,3,3,3,7,2,1,2,2,3,3 fitted to the continuous Rayleigh distribution by the maximum likelihood estimator $\hat{\lambda} = 0.0742$. Using (4) implies the output data are following the discrete Rayleigh distribution with parameter: $\theta = \hat{\theta} = \exp(-\hat{\lambda}) = 0.9285$. Two cases for the prior have been considered:

1. Non informative prior with: $a = 0, b = 1$
2. Informative priors, we use the prior information

- Case (1): $E(\theta) = 0.5$, $V(\theta) = 0.0833$ .This gives $a = b = 1$

- Case(2): $E(\theta) = 0.5$, $V(\theta) = 0.25$ .This gives $a = b = 0.5$

Using the non-informative prior, the Bayesian estimate for θ under SELF is 0.9271, and fixing the value of the GELF shape parameter$M = 1$,the Bayesian estimate is0.9268.When using the informative prior Case (1),the Bayesian estimate is 0.9274 under SELF and 0.9271 under GELF, and for Case(2), the Bayesian estimate is0.9291 under SELF and 0.9289 under GELF 0.95% HPD interval for θ whena = b = 1 is (0.88935,0.955) and when a = b = 0.5is (0.891,0.956) while it is un-attainable when using the non-informative prior.

Prediction for the fifth order observation y_5 is 2.0038 which is identical to the least integer value of the true observation. Assuming $k_0 = k_1 = 1$, we consider testing the following four hypotheses:

1) $H_0: \theta < 0.90$, $H_1: \in \theta \in (0.9,1)$
2) $H_0: \theta < 0.92$, $H_1: \in \theta \in (0.92,1)$
3) $H_0: \theta < 0.93$, $H_1: \in \theta \in (0.93,1)$
4) $H_0: \theta < 0.94$, $H_1: \in \theta \in (0.94,1)$

Based on the test statistic obtained in (37),H_0 is rejected for the first three hypotheses and is not rejected for the fourth hypothesis. Hence the Bayesian decision is to indicate $\theta \in (0.90,0.94)$ which coincides with the numerical estimates of θ. Also, in testing the simple hypothesis: $H_0: \theta = 0.92, H_1: \theta = 0.93$, based on (39) H_0 is rejected. This gives a high liability and efficiency of the Bayesian testing inference as the high performance of the Bayesian estimation procedures.

7.2 Simulation study

For the purpose of illustration and comparison, a simulation study is conducted to compare performance of the Bayesian estimators,θ_S, θ_G with other classical maximum likelihood and moment estimators. The simulation study involves generating 1000 samples with different sample sizes $n = 10,15,20$ from the discrete Rayleigh distribution. Three settings for initial values of the parameter θ are considered ($\theta_0 = 0.3867$, $\theta_0 = 0.6065$, $\theta_0 = 0.9048$). The prior parameters a, b are indicated by fixing the values of $a = 2$,3 and estimating the corresponding values of b assuming θ_0 as the mod of the prior using (15).Under the GELF values of the shape parameter M are given to be 1,2 for each setting of the Bayesian estimation. The following steps describe the simulation process.

1. Generate $U_i, i = 1,2, \dots n$ random numbers from the Uniform(0,1)distribution.
2. Use the relation: $X_i = \left(\frac{-lin\,(1-U_i)}{\lambda_0}\right)^{\frac{1}{2}}$ to generate n random numbers from the continuous Rayleigh distribution with parameter λ_0.
3. Round the values of $X_i, i = 1,2, \dots, n$ to their corresponding least integers $[X_i]$ to have a sample of size n from the discrete Rayleigh distribution with parameter $\theta_0 = e^{-\lambda_0}$.
4. Compute the values of θ_S, θ_G and their squared errors as : $(\hat{\theta}_i - \theta_0)^2$ where $\hat{\theta}_i$ is their corresponding estimate of θ_0.
5. Repeat the steps 1-4 ,1000 times and compute their MSE as

$$MSE(\hat{\theta}_i) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i - \theta_0)^2$$

The Tables 1-3 show the MSE of the Bayesian and non-Bayesian estimators at different settings of the simulation study.

Table 1. The MSE of the estimators when $\theta_0 = 0.3867$

(a,b)	Estimator	$n = 10$	$n = 15$	$n = 20$
	θ_{ml}	0.0048	0.0041	0.0032
	θ_{mom}	0.0062	0.0057	0.0049
(2,2.586)	θ_S	0.0098	0.0074	0.0061
	θ_G ,$M=2$	0.0089	0.0081	0.0072
	θ_G ,$M=3$	0.0094	0.0083	0.0074
(3,2.2976)	θ_S	0.0121	0.0102	0.0097
	θ_G ,$M=2$	0.0100	0.0099	0.0093
	θ_G ,$M=3$	0.0103	0.0101	0.0095

Table 2. The MSE of the estimators when $\theta_0 = 0.6065$

(a,b)	Estimator	$n = 10$	$n = 15$	$n = 20$
	θ_{ml}	0.0057	0.0043	0.0038
	θ_{mom}	0.0071	0.0052	0.0049
(2,1.6488)				

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	θ_S	0.0085	0.0029	0.0022
	θ_G ,$M=2$	0.0064	0.0018	0.0011
	θ_G ,$M=3$	0.0051	0.0013	0.0010
(3,2.2976)	θ_S	0.0014	0.0011	0.0009
	θ_G ,$M=2$	0.0042	0.0013	0.0011
	θ_G ,$M=3$	0.0086	0.0028	0.0015

Table 3. The MSE of the estimators when $\theta_0 = 0.9048$

(a,b)	Estimator	$n=10$	$n=15$	$n=20$
	θ_{ml}	0.0063	0.0054	0.0039
	θ_{mom}	0.0074	0.0059	0.0041
(2,1.1052)	θ_S	0.0061	0.0027	0.0018
	θ_G ,$M=2$	0.0062	0.0013	0.0010
	θ_G ,$M=3$	0.0059	0.0013	0.0009
(3,1.2104)	θ_S	0.0013	0.0010	0.0007
	θ_G ,$M=2$	0.0031	0.0012	0.0009
	θ_G ,$M=3$	0.0041	0.0011	0.0009

As it appears from the tables 1-3 based on their (MSE),as the values of θ_0 increases from 0.3867 to 0.9048 the Bayesian estimators θ_S, θ_G gave a better estimates for the corresponding values of θ_0 than both $\theta_{ml}, \theta_{mom}$ .For relatively small parameter values when $\theta = \theta_0 = 0.3867$,the MLE θ_{ml} dominate other estimators for $n = 10$, but as the sample increases to $n = 20$,the Bayesian estimators are still have approximately a reasonable MSE. It is also clearly that, θ_G (The Bayesian estimator obtained under GELF) gave a better estimate than θ_S(The Bayesian estimator obtained under SELF).The prior parameters a , b have a manifest affection on the performance of the Bayesian estimators,(when $\theta = \theta_0 = 0.3867$ the MSE's of the Bayesian estimators increases as the values of a , $b$increases, and the converse is true when $\theta_0 = 0.9048$).The shape parameter M of the GELF also effect the accuracy of θ_G, when $\theta = \theta_0 = 0.3867$,$\theta_G$ has less MSE when $M = 2$ than when $M = 3$ and when $\theta_0 = 0.9048$ it has less MSE when $M = 3$ than when $M = 2$.

8. Conclusion

In this paper, a general Bayesian inference is investigated for the scale parameter of the discrete Rayleigh distribution. The Bayesian estimation is modified to obtain explicit forms for both point and HPD interval estimators. Numerical results through a real lifetime's data and a simulation approach show a considerable efficiency in the performance of the Bayesian estimators. Hypotheses testing and prediction for future observations are also included and illustrated. Generally the prior parameters and the shape parameter of the GELF have a reasonable affection in the Bayesian inference. This affection is clearly depends on the true values for the indexed parameter and the size of the drawn sample.

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Appendix (1)

Proof of theorem 1:

Set:
$m_1 = \max\ \{X_i, i = 1,2,\dots,n\}$,
$m_2 = min\,\{X_i, i = 1,2,\dots,n\}$
Since $0 < \theta < 1$, this implies

$$(1-\theta^{2m_1+1}) \le (1-\theta^{2X_i+1}) \le (1-\theta^{2m_2+1})$$

$$\prod_{i=1}^{n}(1-\theta^{2m_1+1}) \le \prod_{i=1}^{n}(1-\theta^{2X_i+1}) \le \prod_{i=1}^{n}(1-\theta^{2m_2+1})$$

Substitute $\prod_{i=1}^{n}(1-\theta^{2m_1+1})$ in the numerator of (16) and $\prod_{i=1}^{n}(1-\theta^{2m_2+1})$ in the denominator of (16) and use the binomial expansions:

$$\prod_{i=1}^{n}(1-\theta^{2m_1+1}) = (1-\theta^{2m_1+1})^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\theta^{i(2m_1+1)}$$

$$\prod_{i=1}^{n}(1-\theta^{2m_2+1}) = (1-\theta^{2m_2+1})^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\theta^{i(2m_2+1)}$$

and simplifying, then we have θ_L.
Similarly, Substitute $\prod_{i=1}^{n}(1-\theta^{2m_2+1})$ in the numerator of (16) and$\prod_{i=1}^{n}(1-\theta^{2m_1+1})$ in the denominator of (16) and use the binomial expansions and simplifying to have θ_U.

Proof of theorem 2:

Following the same proof in theorem 1 and substituting in both numerator and denominator of (16).

Appendix (2)

Define:
$incbeta(a,b,c) = \int_0^a x^{b-1}(1-x)^{c-1}dx$, then

$$\int_{c_1}^{c_2}\theta^{\sum_{j=1}^{n}X_j^2+a-1}(1-\theta)^{n+b-1}\,d\theta = \int_0^{c_1}\theta^{\sum_{j=1}^{n}X_j^2+a-1}(1-\theta)^{n+b-1}d\theta - \int_0^{c_1}\theta^{\sum_{j=1}^{n}X_j^2+a-1}(1-\theta)^{n+b-1}d\theta$$

$$= incbeta\left(c_2, \sum_{j=1}^{n} X_j^2 + a - 1, n + b - 1\right) - incbeta\left(c_1, \sum_{j=1}^{n} X_j^2 + a - 1, n + b - 1\right)$$

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