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Extended Kalman Filter and Density – Based Monte Carlo Filter for Discrete time Stochastic Differential Equation

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Abstract:

Nonlinear Stochastic differential equations SDEs with Unknown state variables are nowadays used In analysis of variation in many branches of Sciences .These types of state variables used to be estimated with extended kalman filter. The history of this filter shows that it is inadequate for many SDEs.A density – based Monte Carlo filter (DBMCF) is provided to estimate the unobservable state variables. The performance of EKF and DBMCF are compared in a simulation experiment, and it is shown the DBMCF is more accurate than EKF.

Key words: Stochastic differential equations, discrete time, state variables, extended Kalman filter, density – based Monte Carlo filter.

1 Introduction

Stochastic differential equation (SDEs) are powerful tools in physics, Finantial mathematics, economics, pharmacokinetic and pharmacodynamic modelinys , which can be applied as a realistic method to describe the variations is systems.

Kalman filter (KF in short) is designed to estimate the state Variable concerning a linear model, and extended Kalman filter (EKF in short) is the Linearized version of for the models with nonlinear characteristics. Tanizaki [5] discussed on this fact that, although the higher order nonlinear filters deduced from KF attained less biased filtering estimates than EKF, but filtering estimates obtained from higher order nonlinear filters are still high biased.

On the other hand, density – based Monte Carlo filters (abbreviated as DBMCF) are less biased compare with EKF, since the unobservable state variable can be generated exactly from the nonlinear functions. The main goal of this article is to compare the performances of EKF and DBMCF for the following discrete time SDEs.

$$x_k = g_k(x_{k-1}, n_k) = x_{k-1} + f(x_{k-1}, t_{k-1})(t_k - t_{k-1}) + G(t_{k-1})(w_k - w_{k-1}) \quad (1.1)$$

$$y_k = h_k(x_k, \varepsilon_k) = h\,(x_k, t_k) + \varepsilon_k \quad (1.2)$$

Where $w_j = w_k - w_{k-1}$ is the noise at time t_k , With $w_j = w(t_j)$ a standard Wiener process,$g_k(x_{k-1}, x_k)$ is the function of state transition ,and $h_k(x_k, \varepsilon_k)$ is the function of measurement variable, also y_k is called measurable variable.

The process (ε_k) is white noise with known distribution pdf p (v_k) at each timet_k. Both w (t_k) and ε_k are independent and independent of initial state x_o.This article in organize in three sections.

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In section 2 we discuss on discrete time stochastic differential equations and their state estimations by using EKF and DBMCF. An example with method of simulation is presented in section and these two methods are compared.

2 Discrete time SDEs models and state estimation

Since most of kalman filter applications are implemented in digital computers, it is natural to rewrite the continuous time SDEs.

d x (t)= f(x(t) , t) dt + G(t) dw (t) (2.1)

as (1.1).

Denote by $Y_k = \{y_1, y_2, \dots, y_k\}$ the information set at time t_k.

The conditional expectation .

$X_{k|k} = E(X_K|Y_k)$ is the filtering estimate of state variable X_k , and $X_{k|k-1} = E(X_K|Y_{k-1})$ is the filtering prediction of X_k.

2.1 EKF for discrete time SDEs

In order to apply EKF, the measurement equation and the transition equation are appronimated as the following state – space model:

$$\mathrm{Y_k} \approx \mathrm{h}_{k|k-1} + \mathrm{z}_{k|k-1}(\mathrm{x_k} - \mathrm{x}_{k|k-1}) + \mathrm{s}_{k|k-1}\varepsilon_\mathrm{k} \qquad (2.2)$$

$$\mathrm{X_k} \approx \mathrm{g}_{k|k-1} + \mathrm{T}_{k|k-1}(\mathrm{x_{k-1}} - \mathrm{x}_{k-1|k-1}) + \mathrm{R}_{k|k-1}\eta_\mathrm{k} \qquad (2.3)$$

Where

$$\mathrm{h}_{k|k-1} = (\mathrm{x}_{k|k-1}, 0).$$

$$\mathrm{Z}_{k|k-1} = \left.\frac{\delta\, \mathrm{h_k}(\mathrm{x_k}, \varepsilon_\mathrm{k})}{\delta\, \mathrm{x_k}}\right|_{(\mathrm{x_k},\varepsilon_\mathrm{k})=(\mathrm{x}_{k|k-1},0)}$$

$$\mathrm{S}_{k|k-1} = \left.\frac{\delta\, \mathrm{h_k}(\mathrm{x_k}, \varepsilon_\mathrm{k})}{\delta\, \mathrm{x_k}}\right|_{(\mathrm{x_k},\varepsilon_\mathrm{k})=(\mathrm{x}_{k|k-1},0)}$$

$$\mathrm{g}_{k|k-1} = \mathrm{g_k}(\mathrm{x}_{k-1|k-1}, 0)$$

$$\mathrm{T}_{k|k-1} = \left.\frac{\delta\, \mathrm{g_k}(\mathrm{x_{k-1}}, \eta_\mathrm{k})}{\delta\, \eta_\mathrm{k-1}}\right|_{(\mathrm{x_{k-1}},\mathrm{n_k})=(\mathrm{x}_{k-1|k-1},0)}$$

$$\mathrm{R}_{k|k-1} = \left.\frac{\delta\, \mathrm{g_k}(\mathrm{x_{k-1}}, \eta_\mathrm{k})}{\delta\, \eta_\mathrm{k-1}}\right|_{(\mathrm{x_{k-1}},\mathrm{n_k})=(\mathrm{x}_{k-1|k-1},0)}$$

EKF is given by the following algorithm:

$$\mathrm{X}_{k|k-1} = \mathrm{g}_{k|k-1} \qquad (2.4)$$

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$\sum_{k|k-1} = \mathrm{T}_{k|k-1}^{2} \sum_{k|k-1} + \mathrm{R}_{k|k-1}^{2}\mathrm{Q}_{\mathrm{k}}$ (2.5).

$\mathrm{y}_{k|k-1} = \mathrm{h}_{k|k-1}$ (2.6)

$\mathrm{F}_{k|k-1} = \mathrm{z}_{k|k-1}^{2} \sum_{k|k-1} + \mathrm{s}_{k|k-1}^{2}\ \mathrm{H}_{\mathrm{k}}$ (2.7)

$\mathrm{M}_{k|k-1} = Z_{k|k-1} \sum_{k|k-1}$ (2.8)

$K_k = \mathrm{M}_{k|k-1}\, \mathrm{F}_{k|k-1}^{-1}$ (2.9)

$\sum_{k|k} = \sum_{k|k-1} - \mathrm{K}_{\mathrm{k}}\, \mathrm{F}_{k|k-1}\, \mathrm{K}_{\mathrm{k}}$ (2.10)

$\mathrm{X}_{k|k} = \mathrm{X}_{k|k-1} + \mathrm{K}_{\mathrm{k}}\left(\mathrm{y}_{\mathrm{k}} - \mathrm{y}_{k|k-1}\right)$ (2.11)

Where $\sum_{k|s} = \mathrm{cov}(\mathrm{x}_{\mathrm{k}}|\mathrm{Y}_{\mathrm{s}})$,

$H_k = Var\ (\varepsilon_{\mathrm{k}}), Q_k = Var\left(\eta_{\mathrm{k}}\right)$

The details for EKF Can be found is (4.5) and the reference therein.

2.2 DBMCF for discrete time SDEs

DBMCF is an alternative solution to nonlinear filtering problems, and the resulted algorithm is simple and convenient to compute.

The collection of state vector is denoted by.

$\mathrm{A}_{\mathrm{k}} = \{\mathrm{x}_0, \mathrm{x}_1, \dots, \mathrm{x}_{\mathrm{k}}\}$ (2.12)

Where X_{k} is the value of unobservable variable at time t_{k}, K= 1,2,…, T.

The Joint density function of (Y_k, A_k) is

P (Y_k, A_k) = P $(\mathrm{A}_{\mathrm{k}})$ P $(Y_k|A_k)$. (2.13)

Where P (A_k) and P $(Y_k|A_k)$ are

P (A_k) = P $(X_0) \prod_{s=1}^{k} \mathrm{P}(X_s|X_{s-1})$ (2.14)

P $(Y_k|A_k) = \prod_{s=1}^{k} \mathrm{P}(y_s|X_{s-1})$ (2.15)

Where P $(X_s|X_{s-1})$ and $\mathrm{P}(y_s|X_{s-1})$ are obtained from (2.1) and (2.2) respectively.The filtering density function is given by

$$P = (X_k|Y_k) = \frac{\int P(X_k, A_{k-1}, Y_k)\, dA_{k-1}}{\int P(A_k)\, P(X_k|Y_k)\, dA_k} \quad (2.16)$$

Such that the filtering estimate of the state variable is given by.

$$X_{k|k} = E(X_k|Y_k) = \frac{\int X_k P(Y_k|A_k)\, P(A_k)\, dA_k}{\int P(Y_k|A_k)\, P\,(A_k)\, dA_k} \quad (2.17)$$

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Where dA_k is a shorthand notation for $dx_1\ dx_2 \ldots\ dx_k$. A filtering estimation based on the Monte Carlo technique is given by.

$$X_{k|k} = \frac{\frac{1}{N}\sum_{i=1}^{N} x_{i,k}\ P\ (Y_k|A_{i,k})}{\frac{1}{N}\sum_{i=1}^{N} P\ (Y_k|A_{i,k})} = \frac{\sum_{i=1}^{N} X_{i,k} \prod_{s=1}^{K} P\ (y_s|X_{i,s-1})}{\sum_{i=1}^{N}\prod_{s=1}^{K} P\ (y_s|X_{i,s-1})} \qquad (2.18)$$

Where N is the number simulated paths, and $A_{i,k}$ is the collection of random draws for the ith simulated path, i.e.

$$A_{i,k} = \{x_{i,0}, x_{i,1}, \ldots, x_{i,k}\} \qquad (2.19)$$

Denote

$$W_{j,k} = \frac{\prod_{s=1}^{K} P(y_s|X_{j,s-1})}{\sum_{j=1}^{N}\prod_{s=1}^{K} P(y_s|X_{j,s-1})}$$

Then we have

$$W_{j,k} = \frac{P\ (y_k|\ x_{j,k-1})\ W_{j,k-1}}{\sum_{j=1}^{N} P\ (y_k|\ x_{j,k-1})\ W_{j,k-1}} \qquad (2.20)$$

Where

$$\sum_{j=1}^{N} W_{j,k} = 1$$

And $W_{j,o} = \frac{1}{N}$ The DBMCF of X_k is given by

$$X_{k|k} = \sum_{j=1}^{N} x_{j,k}\ \omega_{j,k} \qquad (2.21)$$

The details of DBMCF can be found in [5].

3 simulation experiment

A one – compartment model with non- linear Absorption and first order elimination is considered, which is used to describe the PK of a drug following an oral dose is [1].

Let Q (t) (mg) be the amount of drug in the gastrointestinal (GI) Trace at time $t\epsilon[0,T]$ (min), which is an unobservable variable.

The suitable stochastic differential equation to explain the processes of absorbation and elimination of the drug is

$$d\ Q\ (t) = -\frac{V_{man}\ \ Q\ (t)}{K_m + Q\ (t)}\ dt + \sigma_q dw(t) \qquad (3.1)$$

where V_{man} (mg / min) is the maximum reaction rate, $K_m(mg)$ is the Michaelis comstant , σ_q is the diffusion parameters of Q (t).

In order to apply EXF and DBMCF for discrete time ,SDE, is approximated with discrete differences is Ito type, i.e.

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$$Q_k = Q_{k-1} - \frac{V_{max}\ Q_{k-1}}{K_m + Q_{k-1}} (t_k - t_{k-1}) + \sigma_q (w_k - w_{k-1}) \quad (3.2)$$

Where $Q_k = Q\ (t_k), W_k = w(t_k), k = 1,2, \dots, T.$

The density – based Monte Carlo filter is given.

$$Q_{k|k} = \sum_{j=1}^{N} Q_{j,k}\ \omega_{j,k} \quad (3.3)$$

Where $Q_{j,k}$ is the simulated value of the unobservable variable at time t_k in the jth path which generated by equation (3.2) directly. $W_{j,k}$ is the weight of the jth path at time t_k, which satisfies $W_{j,0} = \frac{1}{N}$ and $\sum_{j=1}^{N} w_{j\,k} = 1$, $w_{j\,k}$ is calculated with a recursive formula

$$w_{j,k} = \frac{P\ (C_k | C_{k-1}, Q_{j,k-1})\ w_{j,k-1}}{\sum_{j=1}^{N} P\ (C_k | C_{k-1}, Q_{j,k-1})\ w_{j,k-1}} \quad (3.4)$$

Where $P\ (C_k | C_{k-1}, Q_{j,k-1})$ is the conditional density function given by.

$$P\ (C_k | C_{k-1}, Q_{j,k-1}) \frac{1}{\sqrt{2\pi}\,\sigma_k}\ exp \left\{ -\frac{(c_k - m_k)^2}{2\,\sigma_k^2} \right\} \quad (3.5)$$

Where $\sigma_k^2 = \sigma_q^2 (t_k - t_{k-1})$.

In order to compare the performance of EKF and DBMCF for discrete time SDE (3.2), set

$$Q_o = 5\,mg\ c_0 = o\ \frac{mg}{L}\ , V_{max} = 1\frac{mg}{\min}, K_m = 15\,mg\ , \sigma_\varphi^2 = 0.0002$$

and $t = 5, 10, 15, 20, \dots, 400\ min.$

The amounts of drug Q_k are generated from (3.2). The mean absolute error (MAE) is defined as

$$MAE = \frac{1}{N} \sum_{k=1}^{N} |Q_{k-} Q_{k|k}|$$

Which is used to measure of accuracy of estimates , where N is the number of observations.The simulated investigation is repeated 200 times and the MAE of each simulation is calculated.

A quantile analysis is applied to those observed MAEs, and the results are reported is table 1, where DBMCF and EKF, and RD is the relative difference between the values of DBMCF and EKF, i.e.

$$RD = \frac{MAE\ of\ EKF - MAE\ of\ DBMCF}{MAE\ of\ DBMCF}$$

It is found that the 0.95 quantile of MAEs given by DBMCF is smaller than the 0.05 quantile given by EKF. It can be concluded that the errors of estimates given by DBMCF are much smaller that their errors of estimates given by EKF.

Table 1

Quantiles of MAEs of estimated amounts of drug in gastrointestinal tract given by DBMCF and EKF within200 simulations.

Quantiles	0.05	0.30	0.50	0.60	0.70	0.80	0.90	0.95

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DBMCF	0.0233	0.0335	0.0339	0.0418	0.0448	0.0487	0.0549	0.0591
EKF	0.0600	0.0738	0.0797	0.0825	0.0863	0.0908	0.0990	0.1021
RD	1.5747	1.2042	0.9997	0.9735	0.9246	0.8647	0.8117	0.7288

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