

Extended Kalman Filter and Density – Based Monte Carlo Filter for Discrete time Stochastic Differential Equation

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Abstract:

Nonlinear Stochastic differential equations SDEs with Unknown state variables are nowadays used In analysis of variation in many branches of Sciences .These types of state variables used to be estimated with extended kalman filter. The history of this filter shows that it is inadequate for many SDEs.A density – based Monte Carlo filter (DBMCF) is provided to estimate the unobservable state variables. The performance of EKF and DBMCF are compared in a simulation experiment, and it is shown the DBMCF is more accurate than EKF.

Key words: Stochastic differential equations, discrete time, state variables, extended Kalman filter, density – based Monte Carlo filter.

1 Introduction

Stochastic differential equation (SDEs) are powerful tools in physics, Financial mathematics, economics, pharmacokinetic and pharmacodynamic modelinys , which can be applied as a realistic method to describe the variations is systems.

Kalman filter (KF in short) is designed to estimate the state Variable concerning a linear model, and extended Kalman filter (EKF in short) is the Linearized version of for the models with nonlinear characteristics. Tanizaki [5] discussed on this fact that, although the higher order nonlinear filters deduced from KF attained less biased filtering estimates than EKF, but filtering estimates obtained from higher order nonlinear filters are still high biased.

On the other hand, density – based Monte Carlo filters (abbreviated as DBMCF) are less biased compare with EKF, since the unobservable state variable can be generated exactly from the nonlinear functions. The main goal of this article is to compare the performances of EKF and DBMCF for the following discrete time SDEs.

$$x_k = g_k(x_{k-1}, n_k) = x_{k-1} + f(x_{k-1}, t_{k-1})(t_k - t_{k-1}) + G(t_{k-1})(w_k - w_{k-1}) \quad (1.1)$$

$$y_k = h_k(x_k, \varepsilon_k) = h(x_k, t_k) + \varepsilon_k \quad (1.2)$$

Where $w_j = w_k - w_{k-1}$ is the noise at time t_k , With $w_j = w(t_j)$ a standard Wiener process, $g_k(x_{k-1}, x_k)$ is the function of state transition ,and $h_k(x_k, \varepsilon_k)$ is the function of measurement variable, also y_k is called measurable variable.

The process (ε_k) is white noise with known distribution pdf $p(v_k)$ at each time t_k . Both $w(t_k)$ and ε_k are independent and independent of initial state x_0 . This article in organize in three sections.

In section 2 we discuss on discrete time stochastic differential equations and their state estimations by using EKF and DBMCF. An example with method of simulation is presented in section and these two methods are compared.

2 Discrete time SDEs models and state estimation

Since most of kalman filter applications are implemented in digital computers, it is natural to rewrite the continuous time SDEs.

$$d x (t)= f(x(t) , t) dt + G(t) dw (t) \quad (2.1)$$

as (1.1).

Denote by $Y_k = \{y_1, y_2, \dots, y_k\}$ the information set at time t_k .

The conditional expectation .

$X_{k|k} = E(X_k|Y_k)$ is the filtering estimate of state variable X_k , and $X_{k|k-1} = E(X_k|Y_{k-1})$ is the filtering prediction of X_k .

2.1 EKF for discrete time SDEs

In order to apply EKF, the measurement equation and the transition equation are approximated as the following state – space model:

$$Y_k \approx h_{k|k-1} + z_{k|k-1} (x_k - x_{k|k-1}) + s_{k|k-1} \varepsilon_k \quad (2.2)$$

$$X_k \approx g_{k|k-1} + T_{k|k-1} (x_{k-1} - x_{k-1|k-1}) + R_{k|k-1} \eta_k \quad (2.3)$$

Where

$$h_{k|k-1} = (x_{k|k-1}, 0).$$

$$z_{k|k-1} = \left. \frac{\delta h_k(x_k, \varepsilon_k)}{\delta x_k} \right|_{(x_k, \varepsilon_k) = (x_{k|k-1}, 0)}$$

$$s_{k|k-1} = \left. \frac{\delta h_k(x_k, \varepsilon_k)}{\delta \varepsilon_k} \right|_{(x_k, \varepsilon_k) = (x_{k|k-1}, 0)}$$

$$g_{k|k-1} = g_k (x_{k-1|k-1}, 0)$$

$$T_{k|k-1} = \left. \frac{\delta g_k(x_{k-1}, \eta_k)}{\delta x_{k-1}} \right|_{(x_{k-1}, \eta_k) = (x_{k-1|k-1}, 0)}$$

$$R_{k|k-1} = \left. \frac{\delta g_k(x_{k-1}, \eta_k)}{\delta \eta_{k-1}} \right|_{(x_{k-1}, \eta_k) = (x_{k-1|k-1}, 0)}$$

EKF is given by the following algorithm:

$$X_{k|k-1} = g_{k|k-1} \quad (2.4)$$

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$$\Sigma_{k|k-1} = T_{k|k-1}^2 \Sigma_{k|k-1} + R_{k|k-1}^2 Q_k \quad (2.5).$$

$$y_{k|k-1} = h_{k|k-1} \quad (2.6)$$

$$F_{k|k-1} = z_{k|k-1}^2 \Sigma_{k|k-1} + s_{k|k-1}^2 H_k \quad (2.7)$$

$$M_{k|k-1} = Z_{k|k-1} \Sigma_{k|k-1} \quad (2.8)$$

$$K_k = M_{k|k-1} F_{k|k-1}^{-1} \quad (2.9)$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - K_k F_{k|k-1} K_k \quad (2.10)$$

$$X_{k|k} = X_{k|k-1} + K_k (y_k - y_{k|k-1}) \quad (2.11)$$

Where $\Sigma_{k|s} = \text{cov}(x_k | Y_s)$,

$$H_k = \text{Var}(\varepsilon_k), Q_k = \text{Var}(\eta_k)$$

The details for EKF Can be found is (4.5) and the reference therein.

2.2 DBMCF for discrete time SDEs

DBMCF is an alternative solution to nonlinear filtering problems, and the resulted algorithm is simple and convenient to compute.

The collection of state vector is denoted by.

$$A_k = \{x_0, x_1, \dots, x_k\} \quad (2.12)$$

Where X_k is the value of unobservable variable at time t_k , $K= 1,2,\dots, T$.

The Joint density function of (Y_k, A_k) is

$$P(Y_k, A_k) = P(A_k) P(Y_k | A_k) \quad (2.13)$$

Where $P(A_k)$ and $P(Y_k | A_k)$ are

$$P(A_k) = P(X_0) \prod_{s=1}^k P(X_s | X_{s-1}) \quad (2.14)$$

$$P(Y_k | A_k) = \prod_{s=1}^k P(y_s | X_{s-1}) \quad (2.15)$$

Where $P(X_s | X_{s-1})$ and $P(y_s | X_{s-1})$ are obtained from (2.1) and (2.2) respectively. The filtering density function is given by

$$P = (X_k | Y_k) = \frac{\int P(X_k, A_{k-1}, Y_k) dA_{k-1}}{\int P(A_k) P(X_k | Y_k) dA_k} \quad (2.16)$$

Such that the filtering estimate of the state variable is given by.

$$X_{k|k} = E(X_k | Y_k) = \frac{\int X_k P(Y_k | A_k) P(A_k) dA_k}{\int P(Y_k | A_k) P(A_k) dA_k} \quad (2.17)$$

Where dA_k is a shorthand notation for $dx_1 dx_2 \dots dx_k$. A filtering estimation based on the Monte Carlo technique is given by.

$$X_{k|k} = \frac{\frac{1}{N} \sum_{i=1}^N x_{i,k} P(Y_k | A_{i,k})}{\frac{1}{N} \sum_{i=1}^N P(Y_k | A_{i,k})} = \frac{\sum_{i=1}^N X_{i,k} \prod_{s=1}^K P(y_s | X_{i,s-1})}{\sum_{i=1}^N \prod_{s=1}^K P(y_s | X_{i,s-1})} \quad (2.18)$$

Where N is the number simulated paths, and $A_{i,k}$ is the collection of random draws for the i th simulated path, i.e.

$$A_{i,k} = \{x_{i,0}, x_{i,1}, \dots, x_{i,k}\} \quad (2.19)$$

Denote

$$W_{j,k} = \frac{\prod_{s=1}^K P(y_s | X_{j,s-1})}{\sum_{j=1}^N \prod_{s=1}^K P(y_s | X_{j,s-1})}$$

Then we have

$$W_{j,k} = \frac{P(y_k | x_{j,k-1}) W_{j,k-1}}{\sum_{j=1}^N P(y_k | x_{j,k-1}) W_{j,k-1}} \quad (2.20)$$

Where

$$\sum_{j=1}^N W_{j,k} = 1$$

And $W_{j,0} = \frac{1}{N}$ The DBMCF of X_k is given by

$$X_{k|k} = \sum_{j=1}^N x_{j,k} \omega_{j,k} \quad (2.21)$$

The details of DBMCF can be found in [5].

3 simulation experiment

A one – compartment model with non- linear Absorption and first order elimination is considered, which is used to describe the PK of a drug following an oral dose is [1].

Let $Q(t)$ (mg) be the amount of drug in the gastrointestinal (GI) Trace at time $t \in [0, T]$ (min), which is an unobservable variable.

The suitable stochastic differential equation to explain the processes of absorption and elimination of the drug is

$$dQ(t) = - \frac{V_{man} Q(t)}{K_m + Q(t)} dt + \sigma_q dw(t) \quad (3.1)$$

where V_{man} (mg / min) is the maximum reaction rate, K_m (mg) is the Michaelis constant , σ_q is the diffusion parameters of $Q(t)$.

In order to apply EXF and DBMCF for discrete time ,SDE, is approximated with discrete differences is Ito type, i.e.

$$Q_k = Q_{k-1} - \frac{V_{\max} Q_{k-1}}{K_m + Q_{k-1}} (t_k - t_{k-1}) + \sigma_q (w_k - w_{k-1}) \quad (3.2)$$

Where $Q_k = Q(t_k)$, $W_k = w(t_k)$, $k = 1, 2, \dots, T$.

The density – based Monte Carlo filter is given.

$$Q_{k|k} = \sum_{j=1}^N Q_{j,k} \omega_{j,k} \quad (3.3)$$

Where $Q_{j,k}$ is the simulated value of the unobservable variable at time t_k in the j th path which generated by equation (3.2) directly. $W_{j,k}$ is the weight of the j th path at time t_k , which satisfies $W_{j,0} = \frac{1}{N}$ and $\sum_{j=1}^N w_{j,k} = 1$, $w_{j,k}$ is calculated with a recursive formula

$$w_{j,k} = \frac{P(C_k | C_{k-1}, Q_{j,k-1}) w_{j,k-1}}{\sum_{j=1}^N P(C_k | C_{k-1}, Q_{j,k-1}) w_{j,k-1}} \quad (3.4)$$

Where $P(C_k | C_{k-1}, Q_{j,k-1})$ is the conditional density function given by.

$$P(C_k | C_{k-1}, Q_{j,k-1}) = \frac{1}{\sqrt{2\pi} \sigma_k} \exp \left\{ -\frac{(c_k - m_k)^2}{2 \sigma_k^2} \right\} \quad (3.5)$$

Where $\sigma_k^2 = \sigma_q^2 (t_k - t_{k-1})$.

In order to compare the performance of EKF and DBMCF for discrete time SDE (3.2), set

$$Q_0 = 5 \text{ mg}, c_0 = 0 \frac{\text{mg}}{\text{L}}, V_{\max} = 1 \frac{\text{mg}}{\text{min}}, K_m = 15 \text{ mg}, \sigma_q^2 = 0.0002$$

and $t = 5, 10, 15, 20, \dots, 400 \text{ min}$.

The amounts of drug Q_k are generated from (3.2). The mean absolute error (MAE) is defined as

$$MAE = \frac{1}{N} \sum_{k=1}^N |Q_k - Q_{k|k}|$$

Which is used to measure of accuracy of estimates, where N is the number of observations. The simulated investigation is repeated 200 times and the MAE of each simulation is calculated.

A quantile analysis is applied to those observed MAEs, and the results are reported in table 1, where DBMCF and EKF, and RD is the relative difference between the values of DBMCF and EKF, i.e.

$$RD = \frac{MAE \text{ of EKF} - MAE \text{ of DBMCF}}{MAE \text{ of DBMCF}}$$

It is found that the 0.95 quantile of MAEs given by DBMCF is smaller than the 0.05 quantile given by EKF. It can be concluded that the errors of estimates given by DBMCF are much smaller than their errors of estimates given by EKF.

Table 1

Quantiles of MAEs of estimated amounts of drug in gastrointestinal tract given by DBMCF and EKF within 200 simulations.

Quantiles	0.05	0.30	0.50	0.60	0.70	0.80	0.90	0.95
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DBMCF	0.0233	0.0335	0.0339	0.0418	0.0448	0.0487	0.0549	0.0591
EKF	0.0600	0.0738	0.0797	0.0825	0.0863	0.0908	0.0990	0.1021
RD	1.5747	1.2042	0.9997	0.9735	0.9246	0.8647	0.8117	0.7288

References

- [1] N. R. Kristensen, H. Madse, S. H. Ingwersen, Using stochastic differential equations for PH/ PD model development, *J. Pharmacokinetics, Pharmacodynamics*- 32 (1) 2005 109 – 141.
- [2] C. Meza, F. Jaffrezic, J. F. Foulley, REML estimation of variance parameters in nonlinear mixed effects models using the SAEM algorithm, *Biometrical J.* 49 (2007) 876- 888.
- [3] V. Picchini, A. D. Goetano, S. Ditlevsen, Stochastic differential mixed- effects models, *scand.j.stat.*37 (2010)67-90
- [4] D. Simon, optimal state estimation: kalman , hoo and nonlinear approaches , john wiley , sons , newjersey
- [5] H. Tanizaki, Nonlinear Filters, Estimation and Application, springer- Varlay. New York , 1996
- [6] A. Vaart, Efficient maximum likelihood estimation in semi parametric mixture models, *Ann. Stat.* 24 (2) (1996) 862- 878.