Numerical Solution to Duffing Equation Using Hybrid Genetic Algorithm Technique

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ABSTRACT:- In this paper, a heuristic scheme is used to obtain the numerical solution of the non linear ordinary differential equations (NLODEs). The approximate solutions of the given NLODEs are deduced as a linear combination of some log sigmoid basis functions with unknown parameters. The given NLODEs are converted into equivalent global error minimization problems. Genetic algorithm (GA) and a hybrid approach combining GA with Interior point algorithm (IPA) are used to solve the minimization problems and to achieve unknown parameters. The effectiveness of the proposed technique is validated by solving duffing equations. The approximate results are found in good agreement with the exact solutions.

Keywords: Duffing equation, Heuristic Scheme, Nonlinear ordinary differential equation (NLODE), Hybrid Genetic algorithm (HGA), Interior Point Algorithm (IPA).

1. INTRODUCTION

The mathematical modeling of nonlinear oscillators by and large gives rise to nonlinear ordinary differential equations (NLODEs). Due to their potential applications in engineering a deliberated amount of attention has been paid by the researchers to the study of these nonlinear oscillators.

Several approximate analytical and numerical techniques have been investigated and suggested by researchers for solving the NLODEs representing the nonlinear oscillators. For instance, Vahidi et al. [1,2] used restarted adomian decomposition method (RADM) and decomposition method (DM) for solving duffing-van der pol equation. Qian et al. [3] obtained analytical solutions to strongly nonlinear coupled van der pol oscillators by using homotopy analysis method (HAM). Sawoor et al. [4] applied He's variational iteration method (VIM) for solving duffing-van der pol equation. Motsa et al. [5] obtained the solutions of the van der pol and duffing equations with the help of linearisation method (LM). Akbarzade et al. [6] used homotopy perturbation method and variational approach for obtaining analytical solution to nonlinear cubic-quintic duffing oscillator.

Recently, stochastic methods based on artificial neural networks (ANN) and evolutionary computing have gained their importance for the solution of nonlinear problems in engineering and science see for example [7-13] and references therein. For instance, Malik et al. [14, 15] used evolutionary computing scheme based on hybrid genetic algorithm (HGA) for obtaining the numerical solution of duffing van der pol (DVP) oscillator and Troesch's problem. Khan et al. [16, 17] solved van der pol oscillator and Riccati Differential Equation of Arbitrary Order by using HGA approach. Malek et al. [18] solved ODEs using hybrid neural network approach. Arqub et al. [19] applied continuous GA based technique for solving boundary value NLODEs problems.

In this paper, we consider nonlinear duffing equation characterized by the general form as follows [20].

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$$\frac{d^2y}{dt^2}+q\frac{dy}{dt}+q_1y+q_2y^3=g(t) \qquad (1)$$

under the following initial conditions:

$$y(0)=a \quad \text{and} \quad \frac{dy}{dt}(0)=b$$

where q, q_1, q_2, a and b are real constants.

Duffing equation (1) has its importance in various scientific fields like engineering, biology and fluid flow induced vibration [21, 22]. Many numerical methods have been in use for its solution like Tabatabaei et al. [20] made use of differential transform method (DTM), B.Bülbül1 et al. [21] used an improved taylor matrix method (ITMM) and S. Nourazar et.al. [22] applied modified differential transform method (MDTM) for obtaining the numerical solution of duffing equation (1).

In this paper, we investigate the numerical solution of (1) by using a heuristic scheme which is based on the combination of log sigmoid basis functions and evolutionary algorithm (EA).

The forthcoming sections of this paper has been devoted for the following areas: Section 2 is devoted for brief overview of the proposed heuristic scheme. Section 3 gives brief description of Hybrid Genetic Algorithm (HGA) approach. In section 4 results and discussion are presented followed by concluding remarks in section 5.

2. BRIEF OVERVIEW OF HEURISTIC SCHEME

We suppose that the approximate solution $\hat{y}(t)$ of (1) and its first and second derivatives that is $\hat{\dot{y}}(t)$ and $\hat{\ddot{y}}(t)$ are linear combination of some basis functions that can be represented by the following equations:-

$$\hat{y}(t)=\sum_{i=1}^{m}a_i\phi(b_it+c_i)$$

$$=\sum_{i=1}^{m}a_i\left(\frac{1}{1+e^{-(b_it+c_i)}}\right) \qquad (2)$$

$$\hat{\dot{y}}(t)=\sum_{i=1}^{m}a_ib_i\left(\frac{e^{-(b_it+c_i)}}{\left(1+e^{-(b_it+c_i)}\right)^2}\right) \qquad (3)$$

$$\hat{\ddot{y}}(t)=\sum_{i=1}^{m}a_ib_i^2\left(\frac{2e^{-2(b_it+c_i)}}{\left(1+e^{-(b_it+c_i)}\right)^3}-\frac{e^{-(b_it+c_i)}}{\left(1+e^{-(b_it+c_i)}\right)^2}\right) \qquad (4)$$

where $\varphi(t)$ is log sigmoid function defined by

$$\varphi(t)=\frac{1}{1+e^{-t}} \qquad (5)$$

where a_i, b_i, c_i are real valued unknown parameters that needs to be determined and m is the number of basis functions.

To find the unknown parameters (a_i, b_i, c_i), the given NLODE (1) is converted into an equivalent error minimization problem as follows:

$$\varepsilon_j=\varepsilon_1+\varepsilon_2 \qquad (6)$$

where j is the generation / iteration index, ε_1 represents the mean of the sum of square error of the given NLODE (1), ε_2 represents the mean of the sum of square error due to initial conditions of (1) and given by (7) and (8) respectively:

$$\varepsilon_1=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{d^2y}{dt^2}+q\frac{dy}{dt}+q_1y+q_2y^3-g(t)\right)^2 \qquad (7)$$

$$\varepsilon_2=\frac{1}{2}\left(\left(y(0)-a\right)^2+\left(\frac{dy}{dt}(0)-b\right)^2\right) \qquad (8)$$

where N is the total number of steps taken in the solution range [0,1]. The minimization of (6) is performed using evolutionary algorithm such as GA. The optimal values of unknown parameters (a_i, b_i, c_i) are achieved corresponding to the minimum ε_j. Once the

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values of unknown parameters are found, the approximate solution of (1) can be obtained using these unknown values in (2).

3. HYBRID GENETIC ALGORITHM

Genetic algorithm is a stochastic global search EA based on the mechanism of natural selection and natural genetics for finding the global optimum solution for an optimization problem. The algorithm begins by creating a random initial population of individuals called chromosomes. The genetic algorithm uses three main types of rules over this population i.e. Selection, Crossover and Mutation at each step to create the next generation from the current population. These steps are carried out continuously until some defined stoppage criteria is met [7, 9, 14].

Interior-Point Methods (IPM) or barrier methods have gained significant attention towards problem solving whether linear, nonlinear, convex, or non-convex having fundamental applications in engineering optimization problems.IPM moves through the interior of the feasible region towards the optimal solution by making use of newton step or conjugate gradient (CG) step at each iteration to solve Karush-Kuhn-Tucker (KKT) equations in the system [23].

The hybridization approach of evolutionary algorithms with local search methods are more precise, fast convergent, resolves the problem of premature convergence and have the tendency to obtain optimum solution for real world engineering problems [14].

In hybrid GA approach, the chromosome with the best fitness found by GA is fed to the IPA as its starting point which further decreases the error to some satisfactory level [14].

The flow chart of hybrid approach of GA-IPA used in this paper is shown in figure 1.

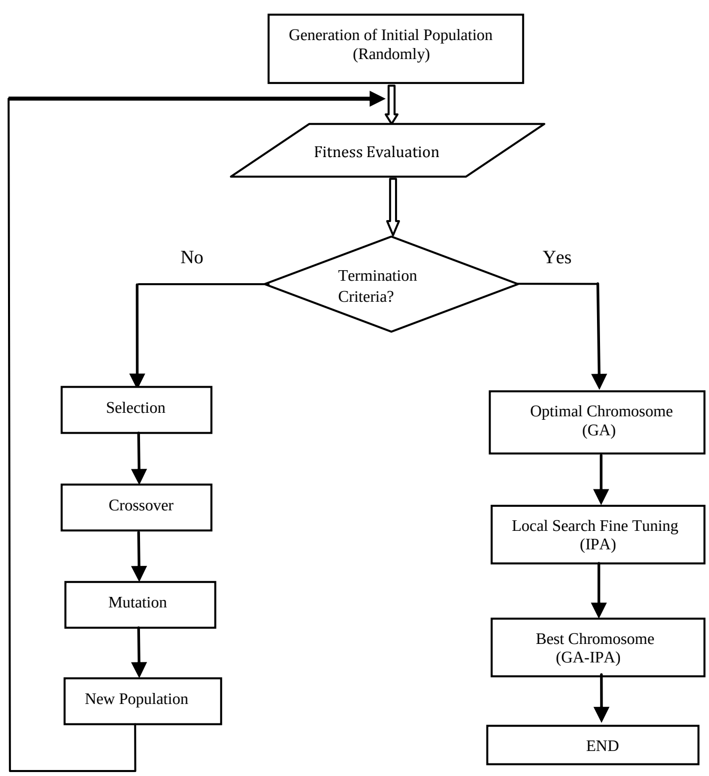

Figure 1:- Flow Chart for Hybrid GA-IPA

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4. RESULTS AND DISCUSSION

In this section, we applied hybrid genetic algorithm to three duffing equations for the validity of the suggested approach. For simulation MATLAB has been used.

Problem 01:- Consider the duffing equation defined as follows [20]:

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}+y+y^3=\cos^3(t)-\sin(t) \qquad (9)$$

with initial values, $y(0)=1$

$$\frac{dy}{dt}(0)=0$$

The exact solution is given by $y(t)=\cos(t)$ [20].

The approximate solution of (9) is obtained in the range [0,1].

To apply the proposed method, we assume that the approximate solution of (9) is given by equation (2) as:

$$\hat{y}(t)=\sum_{i=1}^{10}a_i\phi(b_it+c_i)=\sum_{i=1}^{10}a_i\left(\frac{1}{1+e^{-(b_it+c_i)}}\right) \qquad (10)$$

The fitness function is formulated as follows:

$$\varepsilon_1=\frac{1}{11}\left(\sum_{i=1}^{11}\left(\hat{\ddot{y}}(t)+\hat{\dot{y}}(t)+\hat{y}(t)+(\hat{y}(t))^3-\cos^3(t)+\sin(t)\right)^2\right) \qquad (11)$$

$$\varepsilon_2=\frac{1}{2}\left\{(\hat{y}(0)-1)^2+\left(\hat{\dot{y}}(0)\right)^2\right\} \qquad (12)$$

$$\varepsilon_j=\varepsilon_1+\varepsilon_2 \qquad (13)$$

where $\hat{y}(t),\hat{\dot{y}}(t),\hat{\ddot{y}}(t)$ are given by equations (2) – (4) respectively.

The fitness function given by (13) contains unknown parameters (a_i,b_i,c_i). To achieve these unknown parameters and consequently the approximate solution of (9), GA, IPA and GA-IPA are employed for the minimization of (13). The parameter settings used for the execution of GA and IPA are given in Table 1. We have chosen $m=10$ in (2), therefore total number of unknown parameters i.e. $(a_1,...a_{10},b_1,...b_{10},c_1,...c_{10})$ is equal to 30 and the size of chromosome is also chosen equal to 30. GA and IPA are executed to achieve the minimum fitness ε_j and corresponding values of unknown parameters are obtained. The values of unknown parameters obtained by GA, IPA and GA-IPA are provided in Table 2. These values of unknown parameters are used in (2) that yields the approximate solution $\hat{y}(t)$ of (9). The results achieved by GA, IPA and GA-IPA are provided in Table (3). To show the accuracy of our results, a comparison of absolute errors $|y_{exact}-\hat{y}(t)|$ yielded by our method is made with DTM in Table (4).

Table 1:- Parameter Settings

GA		***IPA***	
Parameter	***Setting***	***Parameter***	***Setting***
Chromosome Size	30	Start Point	randn (1,30)
Population Size	[120 120]	Maximum Iterations	1000
Creation Function	Feasible Population	Maximum Function Evaluations	90,000
Scaling Function	Proportional	SQP Tolerance	1.00E-18
Selection Function	Roulette	X Tolerance	1.00E-18
Mutation Function	Adaptive Feasible	Derivative type	Central Differences
Crossover Function	Heuristic	Hessian	BFGS
Generations	1000	Function Tolerance	1.00E-17

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Function Tolerance	1.00E-17	Non Linear Constraint Tolerance	1.00E-17
Non Linear Constraint Tolerance	1.00E-17	Bounds	[-25, +25]
Bounds	[-25, +25]		

Table 2:- Values of unknown parameters (a_i, b_i, c_i) achieved by the algorithms

Index	*GA*			*IPA*			*GA-IPA*		
(i)	a_i	b_i	c_i	a_i	b_i	c_i	a_i	b_i	c_i
1	0.353815	0.175846	-0.856214	2.117530	1.505805	1.883443	0.351907	0.191572	-0.854980
2	-2.905815	1.267734	-1.755698	-0.633460	-0.145738	-0.013507	-2.914984	1.277960	-1.804955
3	-0.211766	1.569577	-0.843256	-1.207554	-0.101768	0.736005	-0.303373	1.720671	-0.812611
4	0.658063	2.078542	2.664648	1.531343	-1.470851	2.118180	0.774930	2.147336	2.740916
5	-0.809290	1.365109	0.324481	-1.237183	0.017722	1.368671	-0.741044	1.224962	0.345132
6	-0.687705	0.162654	1.037407	-1.727081	-0.685430	-0.401285	-0.704322	0.120908	1.041066
7	2.539299	1.398084	1.013955	0.538060	-1.675019	0.980422	2.542259	1.281255	0.937135
8	-0.656960	-1.229822	0.660971	1.376456	-1.183118	2.133687	-0.686554	-1.175961	0.630322
9	1.357607	-0.745086	1.439071	-0.832967	0.239388	1.481992	1.308688	-0.795701	1.430318
10	-0.877793	-0.101347	2.062736	-2.116517	0.412472	-1.659108	-0.911530	-0.123079	2.058224

Table 3:- Approximate solution by proposed method and comparison with DTM [20] and exact solution.

		Present Method $\hat{y}(t)$			
t	$y_{exact}(t)$	**GA**	**IPA**	**GA-IPA**	**DTM [20]**
0.0	1.000000	1.000032	1.000000	1.000000	---
0.1	0.995004	0.995036	0.995004	0.995004	0.995004
0.2	0.980067	0.980094	0.980066	0.980067	0.980067
0.3	0.955336	0.955357	0.955336	0.955337	0.955336
0.4	0.921061	0.921074	0.921061	0.921061	0.921061
0.5	0.877583	0.877592	0.877582	0.877583	0.877583
0.6	0.825336	0.825343	0.825335	0.825336	0.825336
0.7	0.764842	0.764849	0.764842	0.764842	0.764842
0.8	0.696707	0.696712	0.696706	0.696707	0.696707
0.9	0.621610	0.621611	0.621610	0.621610	0.621610
1.0	0.540302	0.540300	0.540302	0.540302	0.540303

Table 4:- Comparison of absolute errors

	$\lvert y_{exact} - \hat{y}(t) \rvert$			
t	**GA**	**IPA**	**GAIPA**	**DTM [20]**
0.0	3.169E-05	6.911E-09	2.595E-09	---
0.1	3.184E-05	2.940E-08	1.172E-09	1.000E-10

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0.2	2.780E-05	1.272E-07	1.130E-08	5.999E-10
0.3	2.052E-05	1.805E-07	2.490E-08	6.999E-10
0.4	1.344E-05	1.919E-07	1.845E-09	3.999E-10
0.5	9.039E-06	2.127E-07	3.013E-08	5.000E-10
0.6	7.453E-06	2.417E-07	1.636E-08	1.700E-09
0.7	6.898E-06	2.478E-07	2.938E-08	1.060E-08
0.8	5.178E-06	2.393E-07	2.826E-08	3.990E-08
0.9	1.430E-06	2.448E-07	2.251E-08	1.273E-07
1.0	2.795E-06	2.307E-07	5.416E-09	3.599E-07

From the comparison of absolute errors in Table (4), the approximate results by our method are found in a good agreement with the exact solution.

Problem 02:- Consider following duffing equation defined by [20]

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y+8y^3=e^{-3t} \tag{14}$$

with initial values, $y(0)=\frac{1}{2}$

$$\frac{dy}{dt}(0)=-\frac{1}{2}$$

The exact solution is given by $y(t)=\frac{1}{2}e^{-t}$ [20].

The approximate solution of (14) is obtained in the range [0, 1] with a step increment of 0.1 and $m=10$.

The fitness function is formulated as given by:

$$\varepsilon_j=\frac{1}{11}\sum_{i=1}^{11}\left\{\left(\ddot{\hat{y}}(t)+2\dot{\hat{y}}(t)+\hat{y}(t)+8(\hat{y}(t))^3-e^{-3t}\right)^2\right\}+$$

$$\frac{1}{2}\left\{\left(\hat{y}(0)-\frac{1}{2}\right)^2+\left(\dot{\hat{y}}(0)+\frac{1}{2}\right)^2\right\} \tag{15}$$

GA and IPA are implemented using the same settings given in Table 1. The values of unknown parameters corresponding to minimum ε_j are provided in Table 5. These unknown parameters are then used in (2) for obtaining the approximate solution $\hat{y}(t)$ of (14). The results achieved by GA, IPA and GA-IPA are provided in Table (6). To show the accuracy of our results, a comparison of absolute errors $|y_{exact}-\hat{y}(t)|$ yielded by our method is made with DTM in Table (7).

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Table 5:- Values of unknown parameters (a_i, b_i, c_i) achieved by the algorithms

Index	*GA*			*IPA*			*GA-IPA*		
(i)	a_i	b_i	c_i	a_i	b_i	c_i	a_i	b_i	c_i
1	2.293129	0.617222	-0.142444	-1.322203	0.507377	1.348075	2.360455	0.522090	-0.216149
2	0.126528	2.288074	-1.854594	0.635120	-1.158646	-1.921208	0.009034	2.651364	-0.936302
3	1.485096	-0.828495	-0.697257	0.121388	-0.592752	-1.806525	1.690035	-1.149534	-0.735959
4	-0.055276	0.204169	0.708570	-0.002287	-2.643167	-1.311901	0.085961	0.213272	0.713433
5	1.831894	-1.329785	1.179185	-1.275472	0.163188	1.217575	1.860810	-1.063392	1.186541
6	-0.307638	0.658605	0.043081	2.378430	-2.092601	-3.761531	-0.231134	0.684727	0.060463
7	-1.287450	-1.480782	1.492763	-0.080725	-1.009391	-2.536779	-1.243589	-1.322165	1.364485
8	0.178931	-0.608685	0.949201	1.344455	-1.198901	-1.551030	0.314655	-0.578165	0.957718
9	-0.424379	1.584259	-1.453289	-1.432474	0.197816	-3.061420	-0.229890	1.564680	-1.573596
10	-1.421701	2.212946	2.202286	3.220714	0.124315	0.792130	-1.744146	2.280268	3.043006

Table 6:- Approximate solution by proposed method and comparison with DTM [20] and exact solution.

		Present Method $\hat{y}(t)$			
t	$y_{exact}(t)$	GA	IPA	GA-IPA	DTM [20]
0.0	0.500000	0.499917	0.500000	0.500000	---
0.1	0.452419	0.452343	0.452419	0.452419	0.452419
0.2	0.409365	0.409305	0.409365	0.409366	0.409365
0.3	0.370409	0.370367	0.370409	0.370409	0.370409
0.4	0.335160	0.335131	0.335160	0.335160	0.335160
0.5	0.303265	0.303245	0.303265	0.303266	0.303265
0.6	0.274406	0.274393	0.274406	0.274406	0.274406
0.7	0.248293	0.248289	0.248293	0.248293	0.248293
0.8	0.224664	0.224670	0.224665	0.224665	0.224664
0.9	0.203285	0.203296	0.203285	0.203285	0.203285
1.0	0.183940	0.183952	0.183940	0.183940	0.183940

Table 7:- Comparison of absolute errors

	$\lvert y_{exact} - \hat{y}(t) \rvert$			
t	GA	IPA	GAIPA	DTM [20]
0.0	8.324E-05	2.287E-07	4.674E-08	---
0.1	7.619E-05	1.993E-07	1.487E-08	2.000E-10
0.2	6.023E-05	8.188E-08	1.286E-07	3.000E-10
0.3	4.229E-05	1.892E-08	1.464E-07	1.000E-10
0.4	2.882E-05	4.541E-08	1.393E-07	1.000E-10
0.5	2.014E-05	2.592E-08	1.845E-07	3.000E-10

0.6	1.275E-05	1.738E-08	2.422E-07	1.000E-10
0.7	3.950E-06	4.714E-08	2.468E-07	1.000E-10
0.8	5.263E-06	9.933E-08	2.127E-07	0.000E+00
0.9	1.123E-05	1.345E-07	1.987E-07	2.000E-10
1.0	1.241E-05	1.347E-07	2.071E-07	6.000E-10

From Table 7, it is evident that DTM shows better accuracy for this problem but the accuracy of our results can be improved by increasing the number of basic functions.

Problem 03:- Consider the following duffing equation [20]:

$$\frac{d^2y}{dt^2} + 30y = 29\cos(t) \qquad (16)$$

with initial values, $y(0) = 0$

$$\frac{dy}{dt}(0) = 0$$

The approximate solution of (16) is obtained in the range [0, 1] with a step increment of 0.1 and $m = 10$.

The fitness function is formulated as:

$$\varepsilon_j = \frac{1}{11}\sum_{i=1}^{11}\left\{\left(\hat{y}(t) + 30\hat{y}(t) - 29\cos(t)\right)^2\right\} + \frac{1}{2}\left\{(\hat{y}(0))^2 + \left(\hat{y}(0)\right)^2\right\} \qquad (17)$$

GA and IPA are implemented using the same settings given in Table 1. The values of unknown parameters corresponding to minimum ε_j are provided in Table 8. These unknown parameters are then used in (2) that provide the approximate solution $\hat{y}(t)$ of (16). The results achieved by GA, IPA and GA-IPA are provided in Table (9). To show the accuracy of our results, a comparison of absolute errors $\left|y_{exact} - \hat{y}(t)\right|$ yielded by our method is made with DTM in Table (10).

Table 8:- Values of unknown parameters (a_i, b_i, c_i) achieved by the algorithms

Index	*GA*			*IPA*			*GA-IPA*		
(i)	a_i	b_i	c_i	a_i	b_i	c_i	a_i	b_i	c_i
1	2.333663	6.072365	-2.149740	-0.626020	-3.707231	0.834991	5.733757	4.679005	-1.805167
2	-1.559769	-7.062062	9.525698	5.853674	4.798256	-1.354162	-3.839493	-6.524579	8.446476
3	-3.428710	5.275710	-4.554945	-5.473102	5.511391	-4.730017	-9.344167	4.330749	-3.601946
4	0.053740	-24.732021	-4.980504	-2.388092	-0.457899	5.117051	1.544689	-0.461033	-1.119122
5	1.151110	1.835226	-9.215556	6.124702	-5.349461	-1.157321	0.042106	0.216950	-0.526382
6	1.613541	3.361957	3.167418	-1.939147	-8.925945	11.362807	3.725549	-0.442621	4.131786
7	5.704226	-3.398131	-0.822118	4.138324	-0.839011	1.179725	5.590666	-5.205339	-1.164011
8	-1.439488	-6.687608	0.815349	-3.385662	-2.317508	0.436765	-3.403893	-5.061587	0.999128
9	-1.477603	-2.267541	-0.718944	0.826287	-0.254486	-2.704572	0.585082	-0.966148	-1.700010
10	2.808718	5.157137	-20.634516	0.984786	-6.539023	4.264778	0.384147	-1.275899	1.010852

Table 9:- Approximate solution by proposed method and comparison with DTM [20] and exact solution.

t	$y_{exact}(t)$	Present Method $\hat{y}(t)$ GA	IPA	GA-IPA	DTM [20]
0.0	---	0.456575	0.000001	0.000000	---
0.1	0.141291	0.531781	0.141340	0.141292	0.141291
0.2	0.522416	0.731492	0.522517	0.522418	0.522415
0.3	1.027645	0.993403	1.027764	1.027647	1.027600
0.4	1.502173	1.235136	1.502277	1.502175	1.501491
0.5	1.797479	1.376126	1.797537	1.797480	1.792240
0.6	1.814879	1.362712	1.814873	1.814879	1.789126
0.7	1.534517	1.184555	1.534450	1.534515	1.442410
0.8	1.021328	0.875530	1.021214	1.021324	0.762036
0.9	0.406226	0.505914	0.406084	0.406193	-0.198881
1.0	-0.151889	0.163705	-0.152228	-0.152100	-1.365712

Table 10:- Comparison of absolute errors.

t	$\lvert y_{exact} - \hat{y}(t) \rvert$ GA	IPA	GAIPA	DTM [20]
0.0	---	---	---	---
0.1	3.905E-01	4.807E-05	7.957E-07	8.000E-10
0.2	2.091E-01	1.012E-04	1.978E-06	8.630E-07
0.3	3.424E-02	1.192E-04	2.277E-06	4.464E-05
0.4	2.670E-01	1.043E-04	2.016E-06	6.816E-04
0.5	4.214E-01	5.761E-05	1.134E-06	5.239E-03
0.6	4.522E-01	5.817E-06	3.141E-08	2.575E-02
0.7	3.500E-01	6.665E-05	1.635E-06	9.211E-02
0.8	1.458E-01	1.139E-04	4.112E-06	2.593E-01
0.9	9.969E-02	1.415E-04	3.307E-05	6.051E-01
1.0	3.156E-01	3.041E-01	3.040E-01	1.214E+00

From the comparison of absolute errors in Table (10), it is obvious that the absolute errors obtained from the proposed method are smaller than the exact solution.

5. CONCLUDING REMARKS

On the basis of numerical solution and comparison made with the exact solutions and traditional method DTM, it can be concluded that the presented method is viable for solving such nonlinear problems.

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Moreover, the improved performance of the proposed hybrid scheme HGA is quite evident on the basis of absolute errors in all the three problems.

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