The Homotopy Analysis Method for Solving the Sturm-Liouville Eigenvalue Oroblem

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Abstract: In this paper, the homotopy analysis method (HPM) is applied to approximate the eigenvalues of the second order Sturm-Liouville problems. The approximations start with an appropriate initial guesses. The approximate solution for eigenfunction obtains as a series with ħ, λ and x. Substituiting the initial guess of the main problem gives an equation with two parameters ħ which controles the convergence rate and λ which is the unknown eigenvalue. The obtained equation is solved using Newton,s method with a special search algorithm to approximate the eigenvalues. The numerical experiments shows the this method is more accurate than the other methods.

Introduction

The homotopy analysis method (HAM), which first was proposed by Liao in 1992 is a method for solving nonlinear problems. This method has been successfully applied to many nonlinear problems, such as nonlinear differential equations, nonlinear integral equations, partial differential equations, fractional differential equations and so on [3-6].

The HAM does not depend on any small or large parameter. Besides, it logically contains other non-perturbation techniques, such as Adomian decomposition method, Lyapanov artificial small parameter method and δ-expansion method, as proved by Liao [2]. Thus, the HAM is valid for much more nonlinear problems in science and engineering. Moreover, there is a qualitative difference between HAM and other methods that HAM includes an auxiliary parameter $\hbar$ which controls the convergence rate of the HAM series. Abbasbandy et al [7], have shown that auxiliary parameter $\hbar$ plays a basic role in the convergence rate control and prediction of multiple solutions of the equation. They use the auxiliary parameter$\hbar$, for calculating multiple solutions of the Sturm-Liouville problems [1]. We show that the method proposed by Abbasbandy et al [1], is useful for finding just some first eigenvalues of the Sturm-Liouville problems. We use Newton's method to find arbitrary large eigenvalues of the Sturm-Liouville problems accurately.

In this paper we consider the following class of two point eigenvalue problem of the form

$$D[p(x)y'(x)] + \lambda q(x)y(x) = 0, \qquad x \in (0,1),$$

Subject to

$$ay(0) + by'(0) = 0, \quad cy(1) + dy'(1) = 0,$$

Where$a, b, c, d \in \mathbb{R}$, $q(x)$ and $p(x) > 0$, are smooth functions.

Homotopy analysis method

Consider the following nonlinear equation

$$N[u(x)] = 0,$$

Where, N is a nonlinear operator and $u(x)$ is an unknown function. For simplicity, we ignore all initial and boundary conditions, which can be treated in a similar way. By means of the traditional homotopy method, Liao constructs the so-called zero-order deformation equation as follows

$$(1-p)L[\phi(x;p) - u_0(x)] = p\hbar N[\phi(x)],$$

Where, $p \in [0,1]$ is embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(x)$ is an initial guess of $u(x)$ and $\phi(x;p)$ is an unknown function. It is important that we have great freedom to choose auxiliary operator in the HAM. Obviously when $p = 0$ and $p = 1$, the following relations hold respectively

$$\phi(x;0) = u_0(x), \qquad \phi(x;1) = u(x).$$

Thus, as p increases from 0 to 1, the the solution $\phi(x;p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $\phi(x;p)$ in Taylor series with respect to p, we have

$$\phi(x;p) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)p^m, \qquad (3)$$

Where

$$u_m(x) = \frac{1}{m!}\left.\frac{\partial^m \phi(x;p)}{\partial p^m}\right|_{p=0}.$$

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If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are chosen properly, the series (3) converges at $p = 1$, and we have

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x),$$

This must be one of the solutions of the original nonlinear equation as proved by Liao. The governing equation can be deduced from the zero-order deformation equation. Define the vector

$$\vec{u}_n = \{u_0(x), u_1(x), \dots, u_n(x)\}.$$

Differentiating (2), m times with respect to the embedding parameter $p$and setting $p = 0$ and finally dividing by $m!$. We have the so-called mth-order deformation equation

$$L[u_m(x) - \mathcal{X}_m u_{m-1}(x)] = \hbar R_m(\vec{u}_{m-1}(x)),$$

Where

$$R_m(\vec{u}_{m-1}(x)) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1} N[\phi(x;p)]}{\partial p^{m-1}}\right|_{p=0},$$

And

$$\mathcal{X}_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

The Method of Abbasbandy et al [1]

Consider the following eigenvalue problem

$$y''(x) + \lambda y(x) = 0, \quad x \in (0,1), \qquad (6)$$

With the boundary conditions

$$y'(0) = 0, \quad y(1) = 0. \qquad (7)$$

If we solve (6) and (7) analytically we get the eigenvalue and its corresponding eigenfunction as

$$\lambda_k = \left(\frac{2k+1}{2}\right)^2 \pi^2, \quad y_k(x) = \cos\left(\frac{2k+1}{2}\right)\pi x, \quad k = 0,1,2,\dots \qquad (8)$$

Abbasbandy et al [30], have considered the auxiliary linear operator and nonlinear operator respectively as

$$L[\phi(x;p)] = \phi''(x;p) \qquad (9)$$

And

$$N[\phi(x;p)] = \phi''(x;p) + \lambda\phi(x;p). \qquad (10)$$

According to the boundary conditions (7) and the rule of the solution expression (9), we choose the initial approximation in the form $u_0(x) = 1$, and we have the zero-order deformation equation with the initial condition

$$\phi'(0;p) = 0,$$

From (5) and (10) we have

$$R_m(\vec{u}(x;p)) = u''(x;p) + \lambda u(x;p),$$

Now, the solution of mth-order deformation equation (4), for $m \geq 1$ becomes

$$u_m(x) = \mathcal{X}_m u_{m-1}(x) + \hbar \int_0^x \int_0^x R_m(\vec{u}_{m-1}(x)dx\,dx.$$

Consequently, the first few terms of the HAM series solution are as follows

$$u_1(x) = \frac{1}{2}\lambda\hbar x^2,$$
$$u_2(x) = \left(\frac{\lambda\hbar^2}{2} + \frac{\lambda\hbar}{2}\right)x^2 + \frac{1}{24}\lambda^2\hbar^2x^4,$$
$$u_3(x) = \left(\frac{\lambda\hbar^3}{2} + \lambda\hbar^2 + \frac{\lambda\hbar}{2}\right)x^2 + \left(\frac{\lambda^2\hbar^3}{12} + \frac{\lambda^2\hbar^2}{12}\right)x^4 + \frac{1}{720}\lambda^3\hbar^3x^6,$$
$$\vdots$$

Accordingly, the mth-order approximate HAM series solution, $w_m(x)$, is in the form

$$w_m(x) = \sum_{i=0}^{m} u_i(x). \qquad (11)$$

To the mth-order approximatie solution (11), which still depends on the eigenvalue $\lambda$and the auxiliary parameter$\hbar$, condition (8) reads

$$w_m(1) \cong 0. \qquad (12)$$

It can be seen that in the plot of λas a function of $\hbar$ according to equation (12), several horizontal plateaus occur, each of which corresponds to an eigenvalue of the Sturm-Liouville problem. As a first illustration of this uniqueness criterion of the HAM, in Figure 1, λ has been plotted according to equation (12) in the $\hbar$ range $[-2,0]$ and λ range $[2,125]$ for $m = 25$. Four λ-plateaus can be identified in this figure. In Figure 2, which has plotted for the λ range $[125,300]$, the next three λ-plateaus can be identified.

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Table 1 shows the first six eigenvalues and corresponding absolute errors of equation (6) which have been approximated by Abbasbandy et al [30].

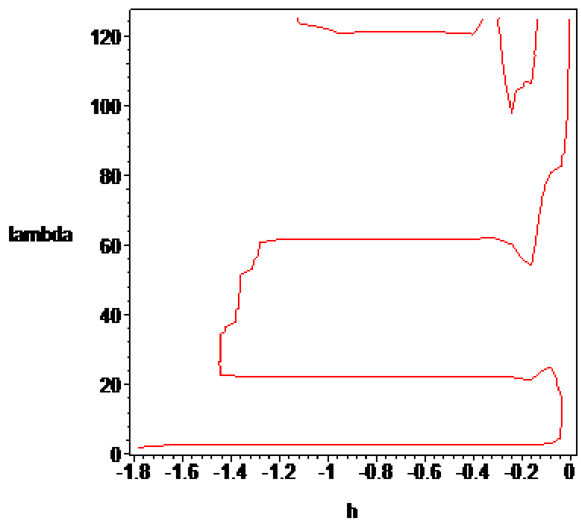

Figure 1: $\hbar$ -curve of approximate solution for λ range [2,125] and $m = 25$.

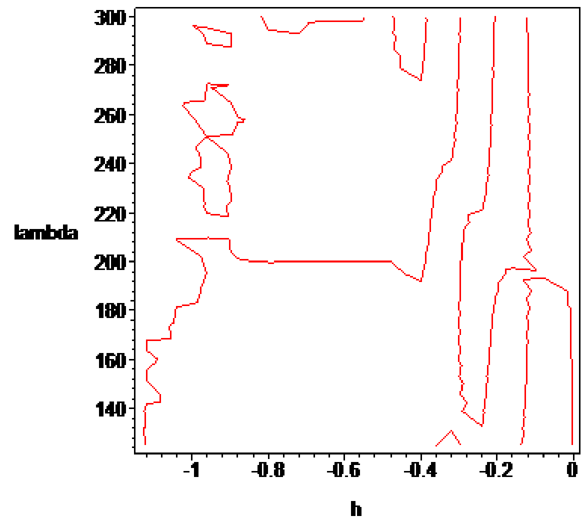

Figure 2: $\hbar$ -curve of approximate solution for λ range [125,300] and $m = 25$.

Table 1: first six eigenvalues and corresponding absolute errors of equation approximated by Abbasbandy et al [30].

Approximated λ_k	Exact λ_k	Absolute error
2.46740110	2.467401100272	2.7E-10
22.20660990	22.206609902451	2.4E-9
61.68502750	61.685027506808	6.8E-9
120.90266801	120.902653913345	1.5E-5
199.85633513	199.859489122060	3.1E-3
298.31364965	298.555533132953	2.4E-1

Newton's Approximation method

To try for approximating some grater eigenvalues of equation (6), we plot the so called $\hbar$-curve for $\lambda \in [300,1000]$, for $M = 25$ in figure 3. As figure 3 shows, it is difficult to identify the horizontal plateaus of λ. Then we conclude that the method proposed by Abbasbandy et al in [30] is not convenient for approximating of large eigenvalues.

In this paper we solve the equation (12) substituting $x = 1$ and $\hbar = -1$ using Newton's iteration method. Table 2 shows approximate eigenvalues and their absolute errors for 15 first eigenvalues of equation (6). We use 30 digits for computations and take $M = 100$.

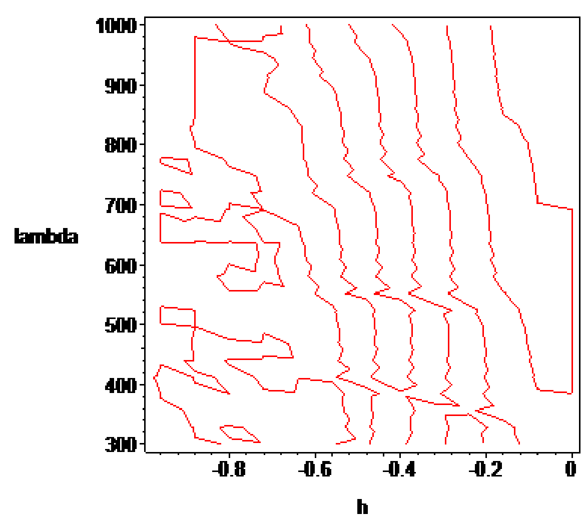

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Figure 3: $\hbar$ -curve of approximate solution for λ range [300, 1000] and $m = 25$.

Table 2: first 15 eigenvalues and corresponding absolute errors of equation [6] approximated in this paper.

k	Approximated λ_k	Exact λ_k	Absolute error
1	2.46740110027233965470862274995	2.467401100272339654708622749969	1.9E-29
2	22.2066099024510568923776047502	22.206609902451056892377604749721	1.7E-26
3	61.6850275068084913677155687612	61.685027506808491367715568749226	1.3E-26
4	120.902653913344643080722515058	120.90265391334464308072251474848	1.3E-24
5	199.859489122059512031398454291	199.85948912205951203139844274749	1.2E-23
6	298.555533132958098219743118750	298.55553313295309821974335274625	2.4E-22
7	416.990785946025401645751285579	416.99078594602540164575724474477	6.0E-21
8	555.165247561276422309229280884	555.16524756127642230944011874303	2.2E-19
9	713.078917978706160198063873364	713.07891797870616021079197474105	1.1E-18
0	890.731797198314615281105157741	890.73179719831461534981281273882	6.8E-17
11	1088.12388522010177950646977992	1088.1238852201017877265026327363	8.2E-15
12	1305.25518204406778525678870977	1305.2551820440676773408614347336	1.1E-13
13	1542.12568767021231220359785175	1542.1256876702122841928892187306	1.2E-13
14	1798.73540209843011123574447169	1798.7354020985356082825859847274	1.1E-10
15	2075.08432533077125108453381958	2075.084325329037649609951732724	1.7E-09

Conclusions

In this paper, the HAM has been applied to numerically approximate the eigenvalues of the Sturm-Liouville problems. In the plot of λ as a function of $\hbar$, several horizontal plateaus occur, which indicates the existence of multiple solutions. Indeed, every horizontal plateaus corresponds to an eigenvalue of the Sturm-Liouville problem. We show that this method is just suitable for approximating some first eigenvalues of the Sturm-Liouville problem, because it is difficult to identify the horizontal plateaus of $\hbar$-curve for large values of $\hbar$. Instead, we use Newton's iteration method with suitable initial values to solve the proposed nonlinear equation. The illustrated example shows the efficiency of this method compared with classic HAM.

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