Recovery of Sparse Signal using Hybrid IRLS-SSF

Hassaan Haider[1*], Jawad Ali Shah[2], Usman Ali[1], Amir A. Khaliq[1], Ijaz Mansoor Qureshi[3]

[1]Department of Electrical Engineering, National University of Science and Technology, College of Electrical and Mechanical Engineering, Rawalpindi, Pakistan
[2]Department of Electronic Engineering, Faculty of Engineering and Technology, International Islamic University Islamabad, Pakistan
[3]Department of Electrical Engineering, Air University, Islamabad, Pakistan

Abstract: In Compressed Sensing (CS), sparse signal recovery is computationally the most exhaustive part, where we have very limited observations. We need supplementary regularization constraints, in order to seek a sparse solution for highly ill-conditioned underdetermined system of linear equations. This paper presents a novel method for reconstructing a sparse signal from limited measurements using Iterative-Reweighted-Least-Squares (IRLS) along with Separable Surrogate Functionals (SSF) algorithm. It is evident from Simulation results, that the estimated signal with the Hybrid IRLS-SSF is far more accurate than the recovered signal, using PCD, IRLS and SSF separately. Finally, the proposed Hybrid IRLS-SSF algorithm efficiency is authenticated experimentally, by almost exact recovery of a K-sparse one-dimensional signal and two-dimensional test image from few random measurements.

Keywords: Sparse Signal Recovery, Compressed Sensing, Iterative-Reweighted-Least-Squares, Iterated shrinkage, Separable Surrogate Functionals.

1. Introduction

In sparse signals, total energy is concentrated in small number of components compared to its dimensions [1]. Mathematically, the signal $\mathbf{x} \in \mathbb{R}^n$ is K-sparse if $\|\mathbf{x}\|_0 \leq K \ll n$, where l_0 quasi-norm can be computed as, $\|\mathbf{x}\|_0 = \#\{j: x_j \neq 0\}$ [2].

In compressed sampling or sensing (CS) theory, sparsity in signals allows us to under sample the signal well below the Nyquist minimum sampling criteria. In CS, fewer elements in sparse signal contain complete information compared to its dimensions, so that exact recovery from fewer measurements is possible [3, 4]. CS has many applications i.e. Magnetic Resonance Imaging (MRI), where acquiring data can be time consuming and costly [5]. MRI acquisition process using CS can reduce the cost of acquisition [4].

In CS, each measurement is obtained by projection of original signal to test function $\boldsymbol{\varphi_i}$ [6]:

$$y_i = <\mathbf{x}, \boldsymbol{\varphi}_i> = \boldsymbol{\varphi}_i{}^{\mathrm{T}}\mathbf{x}, 1 \leq i \leq m \ll n$$

or in matrix form $\mathbf{y} = \boldsymbol{\Phi}\mathbf{x}$ (1)

where measurement or sensing matrix $\boldsymbol{\Phi}: \mathbb{R}^n \to \mathbb{R}^m$ reduces the dimensions to m rows, composed of vectors $\boldsymbol{\varphi}_1{}^{\mathrm{T}}, \boldsymbol{\varphi}_2{}^{\mathrm{T}}, \ldots \boldsymbol{\varphi}_m{}^{\mathrm{T}}$ and observation vector is defined by $\mathbf{y} \in \mathbb{R}^m$.

Restricted Isometry Property (RIP) must be satisfied by measurement matrix $\boldsymbol{\Phi}$ to prevent information in the actual signal from distortion, such as random Bernoulli or Gaussian matrices. If the independent and identically distributed Gaussian matrix satisfies the RIP and there are sufficient measurements taken i.e. $m \geq K \log(n/K)$, then sparse signal could be recovered with high probability [7, 8]; where m measurements are taken from a signal of length n having K non-zero elements.

In system, defined by equation (1), there can be different signals that will give same set of measurements, when $\boldsymbol{\Phi}$ is full ranked. Therefore, finding the sparsest solution is highly ill-conditioned problem [2]. Therefore, regularization constrains must be added for required solution.

The minimum l_2 norm based solution minimizes the total energy of the approximated signal and has a unique solution. However, solution is generally non-sparse.

l_0 norm minimization make use of sparsity constraint for finding an estimate of solution to problem in equation (1) with few nonzero entries. However, it has non-convex formulation for finding the sparse solution and computationally intractable as it involves an exhaustive search $\binom{n}{K}$ for nonzero entries of the reconstructed vector [8].

This paper presents a novel algorithm of merging two popular Iterative-Shrinkage algorithms IRLS and SSF in order to accelerate convergence of objective function. As IRLS converges very fast for initial iterations then slows down significantly afterwards, whereas SSF converges slowly for initial iterations then converges quickly for smaller value of objective function. The main idea in Hybrid IRLS-SSF algorithm is to utilize the strong convergence zones of both i.e. IRLS and SSF algorithm. When IRLS slows down it immediately shifts to SSF algorithm.

The remaining paper has following sections:- Section: 2 has sparse signal recovery algorithms review. Section: 3 describes the IRLS algorithm.

(DOI: dx.doi.org/14.9831/1444-8939.2014/2-6/MAGNT.43)

Section: 4 discusses the SSF algorithm. Section: 5 describes the proposed Hybrid IRLS-SSF algorithm. Section: 6 contains simulation results and Section: 7 contains conclusion based on simulation results.

2. Sparse Signal Recovery Algorithms

Linear regression is the oldest application for sparse signal estimation. Several techniques have been introduced by researchers to recover sparse signal at less computational cost.

There is temptation for l_2 norm minimization given in equation (2) due to its unique solution and computationally tractable. But, in sparse signal recovery problem in equation (1) gives dense solution as it minimizes the energy of a signal, which is distributed over large number of elements in estimated signal.

$$\hat{\mathbf{x}} = \underset{x}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{\Phi x}\|_2^2 = \boldsymbol{\Phi}^{\boldsymbol{T}}(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\boldsymbol{T}})^{-1}\boldsymbol{y} \quad (2)$$

Sparse approximation problems such as l_0 minimization in equation (3) use the sparsity constraint as a regularizer to find an approximate solution to equation (1) with few nonzero entries. But it is non convex and computationally not tractable, and sensitive to noise.

$$\hat{\mathbf{x}} = \underset{x}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{\Phi x}\|_2^2 \text{ subject to } \|\mathbf{x}\|_0 \le K \quad (3)$$

In order to make l_0 norm constraint more tractable, the discontinuous l_0 norm can be replaced by l_p norms where $p \in (0,1]$ or approximated by smooth function such as $\sum_j \log(1+\propto x_j^2)$, $\sum_j x_j^2/(1+\propto x_j^2)$ or $\sum_j (1-\exp(\propto x_j^2))$ [9]. The algorithms from this family of recovery are FOcal Underdetermined System Solver (FOCUSS) algorithm, by Gorodnitsky and Rao [10] and Smoothed l_0 , by Mohimani, Zadeh and Jutten[11] .

The l_1 norm is convex and promotes sparsity in solution, whereas, l_0 norm in equation (3) is generally not tractable and non convex. Thereby, we can replace l_0 norm by l_1 norm to remodel the problem in equation (3) by:

$$\hat{\mathbf{x}} = \underset{x}{\mathbf{argmin}} \|\mathbf{\Phi x} - \mathbf{y}\|_2^2 \text{ subject to } \|\mathbf{x}\|_1 \le \varepsilon \quad (4)$$

where ε can be defined as relaxation positive constant. The convex formulation presented in equation (4) is known as the Least Absolute Shrinkage and Selection Operator (LASSO) [12].

Basis Pursuit (BP) makes use of equality constrained in order to recast the problem as [13]:

$$\hat{\mathbf{x}} = \underset{x}{\mathbf{argmin}} \|\mathbf{x}\|_1 \text{ subject to } \mathbf{y} = \mathbf{\Phi x} \quad (5)$$

Interior point methods and projected gradient may be employed to find solution of convex relaxation problems in equations (4) and (5) numerically.

Pursuit methods use the greedy approach that looks for the best-suited K columns in matrix $\mathbf{\Phi}$ that fits in measurement vector $\mathbf{y}$, and then uses pseudo-inverse to reconstruct the estimate of the signal. Variants of greedy pursuit method are Matching Pursuit (MP), Orthogonal Matching Pursuit (OMP) and Compressive Sampling Matching Pursuit (CoSaMP) [15], etc. Greedy algorithms has faster recovery process when compared to the convex optimization technique.

The *Iterative-Shrinkage* algorithms are another well known method for sparse signal recovery, that minimizes the objective function [16]:

$$\hat{\mathbf{x}} = \underset{x}{\mathbf{argmin}}(\|\mathbf{y} - \mathbf{\Phi x}\|_2^2 + \beta\, \|\mathbf{x}\|_1) \quad (6)$$

The parameter $\beta \ge 0$ is used as regularizer, that defines sparsity in our estimated solution. Larger values of β sparsify the solution whereas smaller values lead to dense solution. Optimization algorithms such as conjugate gradient and steepest-descent are generally not efficient for problem in equation (6), for high dimensional signals frequently encountered in image processing applications [9]. All *Iterative Shrinkage* algorithms involve *Shrinkage* function $\boldsymbol{x}_{opt} = \boldsymbol{S}_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$, which maps the input $\boldsymbol{\alpha}$ to output $\boldsymbol{x}_{opt}$. The values below the threshold T, are mapped to zero ($\boldsymbol{S}_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = \mathbf{0}$ for $|\boldsymbol{\alpha}| < \text{T}$) and those values which are outside this *threshold* $\boldsymbol{T}$ are shrunk [9]. Every iteration in the *Iterative-Shrinkage* algorithms includes a shrinkage step, multiplication by measurement matrix $\mathbf{\Phi}$ along with its transpose ($\boldsymbol{\Phi}^{\boldsymbol{T}}$). Iterative-Shrinkage algorithms are simple in structure still highly efficient in minimizing the problem in equation (6) [9]. Separable Surrogate Functionals (SSF) [17], Fast Iterative Shrinkage Thresholding Algorithm (FISTA) [18], Parallel Coordinate Descent (PCD) [9] and Iterative Reweighted Least Squares (IRLS) algorithm [19] belong to Iterative Shrinkage algorithms.

1. Iterative Reweighted Least Square

Iterative Reweighted Least Square (IRLS) algorithm is used for many optimization problems. IRLS is useful to find maximum likelihood function of generalized linear model, rather, minimizing the simple l_2 norm, IRLS minimizes a weighted residual [19]. IRLS converts a non$-l_2$ norm to l_2 ones using weighting [10]. In equation (6), $\|x\|_1$ can be replaced by $0.5\boldsymbol{x}^T\boldsymbol{W}^{-\mathbf{1}}(\boldsymbol{x})\boldsymbol{x}$, where $\boldsymbol{W}(\boldsymbol{x})$ is diagonal matrix contains $\boldsymbol{W}[k,k] = 0.5\boldsymbol{x}[k]^2/\|\boldsymbol{x}\|_1$ values in its diagonal. We have objective function of the form:

$$f(\boldsymbol{x}) = 0.5\|\boldsymbol{y} - \boldsymbol{\Phi}\|_2^2 + 0.5\beta\boldsymbol{x}^T\boldsymbol{W}^{-1}(\boldsymbol{x})\boldsymbol{x} \quad (7)$$

In order to update solution $\boldsymbol{x}_0$, first we update $\boldsymbol{x}$ assuming fixed $\boldsymbol{W}$ by minimization of quadratic function:

$$\nabla f(\boldsymbol{x}) = -\mathbf{\Phi}^{\mathbf{T}}(\mathbf{y} - \mathbf{\Phi x}) + \beta \boldsymbol{W}^{-1}(\boldsymbol{x})\boldsymbol{x} = 0 \quad (8)$$

Then $\boldsymbol{x}$ can be updated by taking inverse of matrix: $\mathbf{\Phi^T\Phi + \beta W^{-1}}$, afterwards we can update $\mathbf{W}$ based on the new solution.

As simple IRLS performs poorly for high dimensional signal, the above mentioned algorithm was modified by Adeyemi and Davies resulting in another Iterative Shrinkage algorithm by adding and subtracting $c\ .\ \boldsymbol{x}$ from equation (8). The resulting iterative equation becomes:

(DOI: dx.doi.org/14.9831/1444-8939.2014/2-6/MAGNT.43)

$$x_{k+1} = \left(\frac{\beta}{c}W^{-1}(x_k) + I\right)^{-1}\left(\frac{1}{c}\Phi^T y - \frac{1}{c}(\Phi^T\Phi - cI)x_k\right)$$
$$= S.\left(\frac{1}{c}\Phi^T(y - \Phi x_k) + x_k\right)$$

Where S is defined as:

$$S = \left(\frac{\beta}{c}W^{-1}(x_k) + I\right)^{-1} = \left(\frac{\beta}{c}I + W(x_k)\right)^{-1} W(x_k)$$

The diagonal matrix S plays the role of shrinkage on the values of $\frac{1}{c}\Phi^T(y - \Phi x_k) + x_k$. Each entry is calculated by:

$$\frac{0.5x_k[i]^2/\|x\|_1}{\frac{\beta}{c} + 0.5x_k[i]^2/\|x\|_1} = \frac{x_k[i]^2}{\frac{2\beta}{c}\|x\|_1 + x_k[i]^2}$$

For large value of $x_k[i]$, the above factor is one, and for small values of $x_k[i]$ its value tends to zero, similar to the way shrinkage works. Where $c \geq 1$ is the relaxation constant and should be chosen as $c > \lambda_{max}(\Phi^T\Phi)/2$ in order to ensure convergence of matrix S, where λ_{max} defines the maximum eigenvalue of matrix.

IRLS solution should never be initialized by zero, as once the solution entry becomes zero, it can never be revived; consequently, IRLS may get stuck in local minima, without reaching global minima.

3. Separable Surrogate Functionals (SSF)

SSF algorithm is another algorithm from family of *Iterative-Shrinkage* algorithms, which was developed by Daubechies, Defrise and De-Mol [17] by adding the following function to objective function in equation (6).

$$g(x, x_0) = \frac{c}{2}\|x - x_0\|_2^2 - \frac{1}{2}\|\Phi x - \Phi x_0\|_2^2 \quad (10)$$

Where choice of c is made to make function $g(.)$ strictly convex, and its Hessian is positive-definite, i.e. $cI - \Phi^T\Phi > 0$, which can be satisfied, if $c > \|\Phi^T\Phi\|_2 = \lambda_{max}(\Phi^T\Phi)$. Recently, Combettes and Wajs [20] have shown that $c > 0.5\lambda_{max}(\Phi^T\Phi)$ for guaranteed convergence of SSF algorithm.

By adding equation (10) to equation (6), the objective function can be rewritten as:

$$h(x) = \frac{1}{2}\|y - \Phi x\|_2^2 + \beta\|x\|_1)$$
$$+\frac{c}{2}\|x - x_0\|_2^2 - \frac{1}{2}\|\Phi x - \Phi x_0\|_2^2 \quad (11)$$

The equation (11) defines the surrogate function that is used in SSF algorithm. The equation (11) can further be simplified as:

$$h(x) = \frac{1}{2}\|y\|_2^2 + \frac{1}{2}\|\Phi x_0\|_2^2 + \frac{c}{2}\|x_0\|_2^2 - y^T\Phi x$$
$$+\beta\|x\|_1 + \frac{c}{2}\|x\|_2^2 - cx^T x_0 + x^T\Phi^T\Phi x_0 \quad (12)$$

$$h(x) = c_1 - x^T[\Phi^T(y - \Phi x_0) + cx_0]$$
$$+\beta\|x\|_1 + \frac{c}{2}\|x\|_2^2 \quad (13)$$

where c_1 is the constant containing terms depending on y and x_0 only.

Let $u_0 = \frac{1}{c}\Phi^T(y - \Phi x_0) + x_0$ and put it in equation (13), then new objective function can be written as:

$$h(x) = c_2 - x^T u_0 + \frac{\beta}{c}\|x\|_1 + \frac{1}{2}\|x\|_2^2 \quad (14)$$

$$h(x) = c_3 + \frac{\beta}{c}\|x\|_1 + \frac{1}{2}\|x - u_0\|_2^2$$

Finally, optimal solution can be obtained using *Shrinkage* function given below:

$$x_{opt} = S(u_0) = S\left(\frac{1}{c}\Phi^T(y - \Phi x_0) + x_0\right) \quad (15)$$

The main objective for using surrogate function is to minimize the objective function iteratively. The sequence of solutions, produced by iterative functions, is proven to converge to the global minima of original function in equation (6). The iterative process can be written as:

$$x_{k+1} = S\left(\frac{1}{c}\Phi^T(y - \Phi x_k) + x_k\right) \quad (16)$$

The above method can be deduced to proximal point algorithm in optimization theory. The function $g(x, x_0)$ measures the distance from previous solutions. The directions $x - x_0$, that are near null-space of Φ, the distance is approximately Euclidean, $c\|x - x_0\|_2^2/2$ and the directions that are spanned by Φ, the distance is close to zero [9]. Therefore, the function $g(x, x_0)$ limits the allowed solutions, which help us in achieving our goal of minimizing the objective function in equation (6) [9].

4. The Proposed Algorithm

The proposed Hybrid IRLS-SSF algorithm is an addition to rich family of *Iterative-Shrinkage* algorithms to solve unconstrained optimization problem in equation (6). The main idea of Hybrid IRLS-SSF algorithm is to combine IRLS and SSF algorithms in such a way that the best convergence region of objective function by both algorithms is utilized. The IRLS algorithm converges quickly for initial few iterations but held up afterwards in local minima without reaching closer to optimal solution. On the other hand, SSF algorithm is initially slow to minimize the objective function, then speeds up its convergence while getting closer to the optimal solution.

The Hybrid IRLS-SSF algorithm starts with IRLS algorithm for fast convergence of objective function in equation (6), till the point IRLS algorithm does not get stuck. The moment IRLS algorithm is held up, Hybrid IRLS-SSF algorithm switches to SSF algorithm, which minimizes the objective function closer to the optimal solution in far less number of iterations than IRLS or SSF alone. Hybrid IRLS-SSF algorithm targets to achieve much better results in terms of mean square error, at less computational cost. Figure-1 describes the proposed Hybrid IRLS-SSF algorithm in detail.

(DOI: dx.doi.org/14.9831/1444-8939.2014/2-6/MAGNT.43)

Task: *Find the value of* $\boldsymbol{x}$ *that minimizes objective function*:

$$f(x) = \underset{x}{\mathbf{argmin}}(\|\mathbf{y} - \mathbf{\Phi x}\|_2^2 + \beta \, \|\mathbf{x}\|_1)$$

Inputs:
Dictionary matrix $\boldsymbol{\Phi} \epsilon R^{m\times n}$,
Compressed measurement $\boldsymbol{y} \in R^m$,
IRLS & SSF thresholding parameter β.

Output:
$K -$ *sparse solution vector* $\boldsymbol{x} \in R^n$

1) **Initialization:**
   Initialize IRLS iteration number $k = 0$ *and SSF iteration number* $j = 0$.
   Initial solution: $\boldsymbol{x}_0 = [x_1, x_2, \dots \dots \dots x_n]$, *where each entry* $x_i \neq 0$.
   Initial residual: $\boldsymbol{r_0} = \boldsymbol{y} - \boldsymbol{\Phi x}$.

2) **IRLS Main Iteration:**
   Increment k *by* 1, *and apply following steps*
   - **2a) Back Projection:** *Compute* $\boldsymbol{e} = \boldsymbol{\Phi^T} \boldsymbol{r}_{k-1}$.
   - **2b) Shrink Update:** Compute the diagonal matrix $\boldsymbol{W}$ by $W[i,i] = \frac{x_k[i]^2}{\frac{2\beta}{c}\|x\|_1 + x_k[i]^2}$
   - **2c) Shrinkage:** Compute $\boldsymbol{e}_s = \boldsymbol{S}(\boldsymbol{x}_{k-1} + \frac{e}{c})$
   - **2d) Update Function Value:** Compute $\boldsymbol{f}(\boldsymbol{x_k}) = \boldsymbol{f}(\boldsymbol{x}_{k-1} - (\boldsymbol{x_s} - \boldsymbol{x}_{k-1})$
   - **2e) SSF Switching Condition:** *If* $|f(\boldsymbol{x_k}) - f(\boldsymbol{x}_{k-1})| < \varepsilon$ *(where constant* $\varepsilon \to 0$) *go to* ***step* 4**, *else go to* ***step* 2*f***.
   - **2f) SSF First Iteration Value:** *Set* $j = k$
   - **2g) Update Solution:** *Compute* $\boldsymbol{x_k} = \boldsymbol{x_{k-1}} + (\boldsymbol{e_s} - \boldsymbol{x_{k-1}})$
   - **2h) Update Residual:** *Compute* $\boldsymbol{r_k} = \boldsymbol{y} - \boldsymbol{\Phi x_k}$
   - **2i) Stopping Rule:** *Go to step* (2) *until, either when* $\|\boldsymbol{x_k} - \boldsymbol{x_{k-1}}\|_2^2 <$ *threshold value or maximum iterations criteria is met.*

3) **Output:** *The final value of* $\boldsymbol{x_k}$

4) **SSF Main Iteration:**
   Increment j *by* 1, *and apply following steps*:
   - **4a) Back Projection:** *Compute* $\boldsymbol{e} = \boldsymbol{\Phi^T} \boldsymbol{x}_{j-1}$
   - **4b) Shrinkage:** *Compute* $\boldsymbol{x_s} = \boldsymbol{S}\left(\boldsymbol{x}_{j-1} + \frac{e}{c}\right)$ *with threshold* β
   - **4c) Update Function Value:** $\boldsymbol{f}(\boldsymbol{x_j}) = \boldsymbol{f}(\boldsymbol{x}_{j-1}(\boldsymbol{x_s} - \boldsymbol{x}_{j-1}))$
   - **4d) Update Solution:** *Compute* $\boldsymbol{x_j} = \boldsymbol{x_{j-1}} + (\boldsymbol{e_s} - \boldsymbol{x_{j-1}})$
   - **4e) Update Residual:** *Compute* $\boldsymbol{r_j} = \boldsymbol{y} - \boldsymbol{\Phi x_j}$
   - **4f) Stopping Rule:** *Go to step* (2) *until, either* $\|\boldsymbol{x_j} - \boldsymbol{x_{j-1}}\|_2^2 <$ *threshold value or maximum iterations criteria is met.*

5) **Output:** *The final value of* $\boldsymbol{x_j}$

Figure-1: The definition of proposed Hybrid IRLS-SSF algorithm

5. Simulation Results and Discussions

This paper uses random Gaussian sensing matrix $\mathbf{\Phi} \in \mathbb{R}^{256\times 512}$, where $m = 256$ represents the number of measurements, as rows and $n = 512$ is the length of the sparse signal, as columns of sensing matrix. This sensing matrix is produced by Gram-Schmidt procedure, where $n \times n$ matrix, having $(-1,1)$ random values and then extracting first m rows of this orthonormal matrix.

The proposed Hybrid IRLS-SSF algorithm was tested using one-dimensional random signal $\boldsymbol{x_0} \in \mathbb{R}^{512}$ with sparsity: $K = 85$, having randomly generated support and random amplitudes for sparse signal recovery. The measurement vector $\boldsymbol{y} = \boldsymbol{\Phi} \, \boldsymbol{x_0} \in \mathbb{R}^{256}$ is produced by compressive sampling of sparse signal $\boldsymbol{x}_0$. Then Hybrid IRLS-SSF algorithm was also tested using a two-dimensional sparse test image with dimensions $\boldsymbol{X} \in \mathbb{R}^{32\times 32}$. The parameter (β) is chosen as 0.001 for IRLS, SSF and PCD algorithms.

The performance of the proposed Hybrid IRLS-SSF algorithm was also compared with IRLS, SSF and PCD based on the normalized mean square error (MSE) and objective function minimization in equation (6), which is computed at each run of 250 iterations by using:

$$\frac{\|\hat{\mathbf{x}}_j - \mathbf{x_0}\|_2^2}{\|\mathbf{x_0}\|_2^2} \;, \; j = 1,2, \dots .250 \qquad (17)$$

Figure-2 shows, how the proposed Hybrid IRLS-SSF algorithm accelerates the convergence of objective function compared to IRLS, SSF and PCD for one-dimensional random sparse signal. Initially, Hybrid IRLS-SSF follows IRLS. However, once in IRLS algorithm SSF switching criterion is met, the Hybrid IRLS-SSF algorithm immediately shifts to SSF and converges more rapidly as compared to IRLS and SSF algorithms.

Figure-3 shows the comparison in performance of the proposed Hybrid IRLS-SSF with IRLS, SSF and PCD for one-dimensional random sparse signal. The performance measure, used in this figure, is MSE, using equation (17). It was observed that initially MSE of Hybrid IRLS-SSF algorithm is the same as that of IRLS algorithm. When Hybrid IRLS-SSF switches to SSF algorithm, MSE of Hybrid IRLS-SSF algorithm drops rapidly. It can be seen that the Hybrid IRLS-SSF algorithm outperforms IRLS, SSF and PCD algorithms, and achieves much better MSE in less number of iterations.

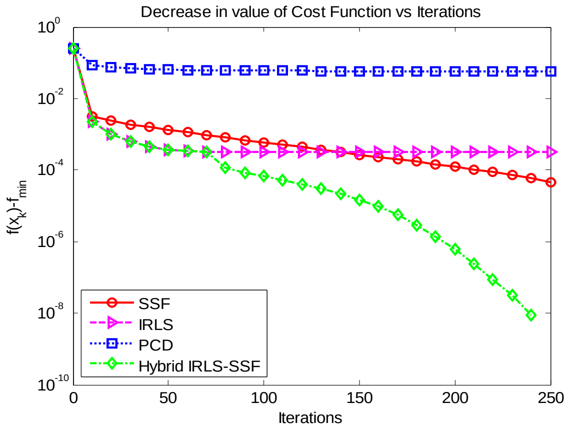

Figure-2: Improvement in convergence w.r.t. minimization of objective function

(DOI: dx.doi.org/14.9831/1444-8939.2014/2-6/MAGNT.43)

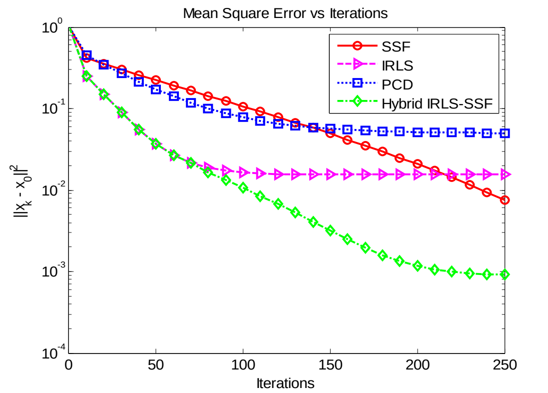

Figure-3: MSE based comparison of IRLS, SSF, PCD and Hybrid IRLS-SSF

Table-1 presents the summary of performance comparison of Hybrid IRLS-SSF algorithm with SSF, IRLS and PCD, when applied to one-dimensional signal. Based on performance measures such as, mean square error, minimum function value and correlation achieved by tested algorithms after 250 iterations, shows that the recovery through proposed algorithm is more precise.

Table-1: Parameters achieved by SSF, IRLS, PCD and Proposed Hybrid IRLS-SSF

Algorithm	*Min Function Value*	*MSE*	*Correlation*
SSF	0.007451	0.007557	0.9973
IRLS	0.007734	0.015516	0.9941
PCD	0.066287	0.050103	0.8417
Hybrid IRLS-SSF	***0.007405***	***0.000915***	***0.9998***

Figure 4 shows the signal amplitude values by IRLS algorithm, compared to original signal. Figure 5 shows the values of recovered signal, using SSF algorithm only. Figure 6 shows the reconstructed signal using PCD comparison with original signal. Figure 7 compares the signal approximation by proposed Hybrid IRLS-SSF algorithm indicating that both the signal amplitude values as well as the support are recovered faithfully with high accuracy.

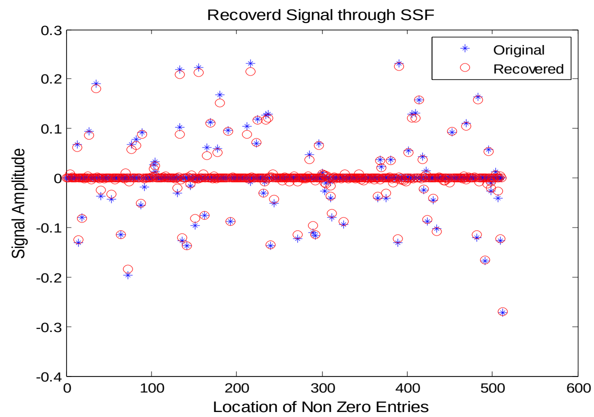

Figure-5: Signal reconstruction through SSF Algorithm

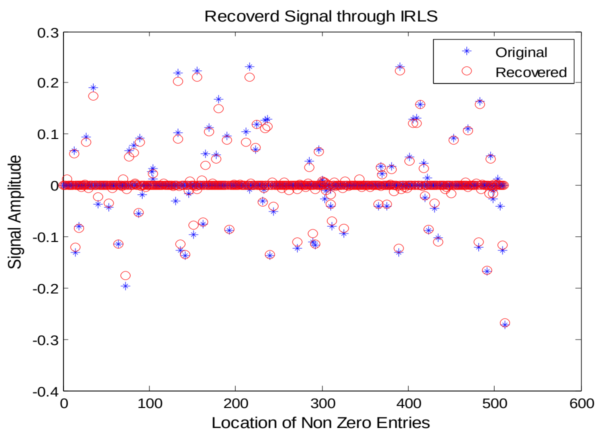

Figure-4: Signal reconstruction through IRLS algorithm

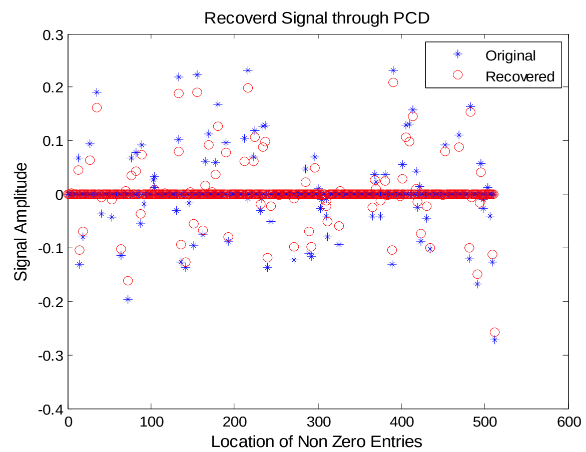

Figure-6: Signal reconstruction through PCD Algorithm

(DOI: dx.doi.org/14.9831/1444-8939.2014/2-6/MAGNT.43)

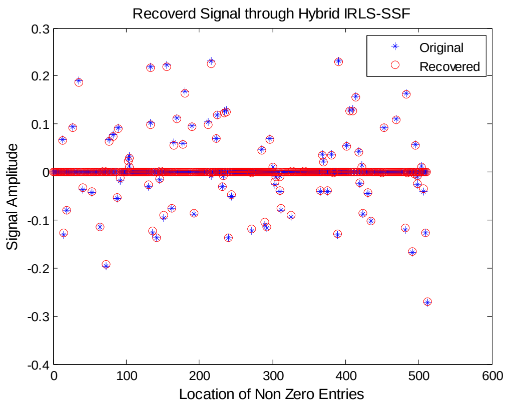

Figure-7: Signal reconstruction with the proposed Hybrid IRLS-SSF algorithm

The Hybrid IRLS-SSF algorithm was also tested to recover two-dimensional test image and compared with IRLS, SSF and PCD algorithms. It proved to be as effective in recovering the two-dimensional image, as it was in one-dimensional signal not only the pixel's value but also the location of the pixel in an image.

Figure-8 shows how the proposed Hybrid IRLS-SSF algorithm speeds up the convergence of objective function, compared to IRLS, SSF and PCD for two-dimensional sparse image. As expected, initially, the Hybrid IRLS-SSF follows IRLS, but once in IRLS algorithm fails to converge further, it switches to SSF algorithm immediately. Hybrid IRLS-SSF convergence is much faster in comparison to IRLS, SSF and PCD algorithms.

Figure-9 shows the performance comparison of the proposed Hybrid IRLS-SSF with IRLS, SSF and PCD for two-dimensional test sparse image. The performance measure used in this figure is MSE, that is evaluated at each iteration using equation (17). It can be observed that, initially MSE of Hybrid IRLS-SSF algorithm is the same as that of IRLS algorithm. When Hybrid IRLS-SSF switches to SSF algorithm, MSE of Hybrid IRLS-SSF algorithm drops rapidly. It can be seen that the Hybrid IRLS-SSF algorithm outperforms IRLS, SSF and PCD algorithms, and achieves much better MSE at less number of iterations.

Figure-10 shows the recovered images from SSF, IRLS, PCD and Hybrid IRLS-SSF. The results show that Hybrid IRLS-SSF was able to recover image with greater accuracy than SSF, IRLS and PCD algorithms.

Table-2 summarizes the performance comparison of Hybrid IRLS-SSF algorithm with SSF, IRLS and PCD, when applied to two-dimensional image, based on performance measures such as: mean square error, minimum objective function value and correlation achieved by tested algorithms after 250 iterations, shows that the proposed algorithm can recover the test image with greater precision.

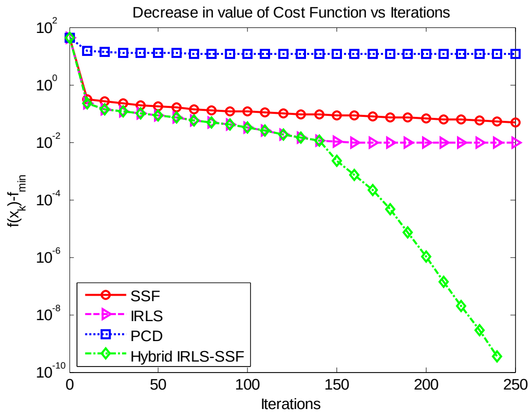

Figure-8: Improvement in convergence of objective function

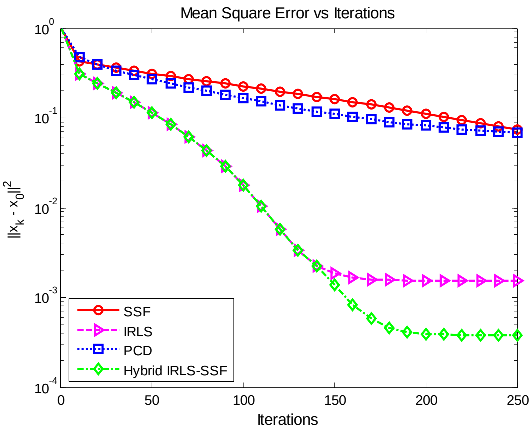

Figure-9: Reduction in MSE using Hybrid IRLS-SSF algorithm

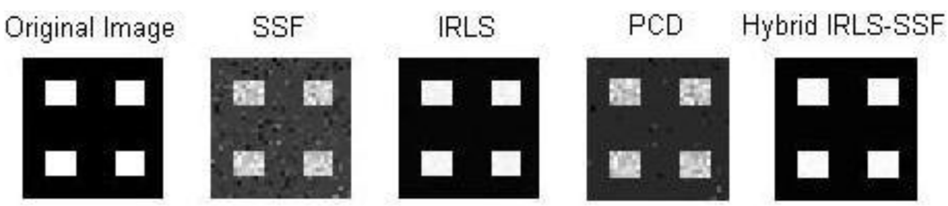

Figure-10: Recovered Images from SSF, IRLS, PCD and Hybrid IRLS-SSF algorithm

Table-2: Parameters achieved by SSF, IRLS, PCD and Proposed Hybrid IRLS-SSF

Algorithm	*Min Function Value*	*MSE*	*Correlation*
SSF	1.2162	0.073519	0.96225
IRLS	1.1760	0.001527	0.99979
PCD	1.3955	0.379500	0.76360

(DOI: dx.doi.org/14.9831/1444-8939.2014/2-6/MAGNT.43)

Hybrid IRLS-SSF	***1.1665***	***0.000380***	***0.99995***

6. Conclusion

Recovering sparse signal from limited observations is facing a challenge of slow convergence. Using Hybrid IRLS-SSF could be very helpful, not only in fast recovery of the original signal but it can also improve the precision of recovered signals, in terms of their amplitude and support, when compared to original sparse signal. The comparison of Hybrid IRLS-SSF algorithm with IRLS, SSF and PCD algorithms has proven that Hybrid IRLS-SSF is far better in terms of speed of recovery and accuracy of recovering support and amplitudes. The proposed Hybrid IRLS-SSF algorithm was able to reconstruct the tested signal and image faithfully at less computational cost.

Acknowledgement:

We would like to thank NUST (CEME) for supporting this research work by providing grants for carrying out research and publishing the research work.

References

[1] Needell, Deanna, Joel Tropp, and Roman Vershynin. "Greedy signal recovery review." In *Signals, Systems and Computers, 2008 42nd Asilomar Conference on*, pp. 1048-1050. IEEE, (2008).

[2] Tropp, Joel A., and Stephen J. Wright. "Computational methods for sparse solution of linear inverse problems." *Proceedings of the IEEE* 98, no. 6 (2010): 948-958.

[3] Candes, Emmanuel J., Justin K. Romberg, and Terence Tao. "Stable signal recovery from incomplete and inaccurate measurements." *Communications on pure and applied mathematics* 59, no. 8 (2006): 1207-1223.

[4] Candès, Emmanuel J., and Michael B. Wakin. "An introduction to compressive sampling." *Signal Processing Magazine, IEEE* 25, no. 2 (2008): 21-30.

[5] Lustig, Michael, David L. Donoho, Juan M. Santos, and John M. Pauly. "Compressed sensing MRI." *Signal Processing Magazine, IEEE* 25, no. 2 (2008): 72-82.

[6] Romberg, Justin. "Imaging via compressive sampling [introduction to compressive sampling and recovery via convex programming]." *IEEE Signal Processing Magazine* 25, no. 2 (2008): 14-20.

[7] Mendelson, Shahar, Alain Pajor, and Nicole Tomczak-Jaegermann. "Uniform uncertainty principle for Bernoulli and subgaussian ensembles." *Constructive Approximation* 28, no. 3 (2008): 277-289.

[8] Baraniuk, Richard G. "Compressive sensing." *IEEE signal processing magazine* 24, no. 4 (2007).

[9] Elad, Michael. *Sparse and redundant representations: from theory to applications in signal and image processing*. Springer, 2010.

[10] Gorodnitsky, Irina F., and Bhaskar D. Rao. "Sparse signal reconstruction from limited data using FOCUSS: A re-weighted minimum norm algorithm." *Signal Processing, IEEE Transactions on* 45, no. 3 (1997): 600-616.

[11] Mohimani, G. Hosein, Massoud Babaie-Zadeh, and Christian Jutten. "Fast Sparse Representation based on Smoothed ℓ0 Norm." In *Independent Component Analysis and Signal Separation*, pp. 389-396. Springer Berlin Heidelberg, 2007.

[12] Wipf, David P., and Bhaskar D. Rao. "Sparse Bayesian learning for basis selection." *Signal Processing, IEEE Transactions on* 52, no. 8 (2004): 2153-2164.

[13] Chen, Scott Shaobing, David L. Donoho, and Michael A. Saunders. "Atomic decomposition by basis pursuit." *SIAM journal on scientific computing* 20, no. 1 (1998): 33-61.

[14] Needell, Deanna, and Roman Vershynin. "Uniform uncertainty principle and signal recovery via regularized orthogonal matching pursuit." *Foundations of computational mathematics* 9, no. 3 (2009): 317-334.

[15] Needell, Deanna, and Joel A. Tropp. "Cosamp: iterative signal recovery from incomplete and inaccurate samples." *Communications of the ACM* 53, no. 12 (2010): 93-100.

[16] Elad, Michael, Boaz Matalon, Joseph Shtok, and Michael Zibulevsky. "A wide-angle view at iterated shrinkage algorithms." In *Optical Engineering+ Applications*, pp. 670102-670102. International Society for Optics and Photonics, 2007.

[17] Daubechies, Ingrid, Michel Defrise, and Christine De Mol. "An iterative thresholding algorithm for linear inverse problems with a sparsity constraint." *Communications on pure and applied mathematics* 57, no. 11 (2004): 1413-1457.

[18] Beck, Amir, and Marc Teboulle. "A fast iterative shrinkage thresholding algorithm for linear inverse problems." *SIAM Journal on Imaging Sciences* 2, no. 1 (2009): 183-202.

[19] Adeyemi, Tony, and M. E. Davies. "Sparse representations of images using overcomplete complex wavelets." In *Proc. IEEE SP 13th Workshop Statistical Signal Processing*, pp. 17-20. 2006.

[20] Combettes, Patrick L., and Valérie R. Wajs. "Signal recovery by proximal forward-backward splitting." *Multiscale Modeling & Simulation* 4, no. 4 (2005): 1168-1200.

(DOI: dx.doi.org/14.9831/1444-8939.2014/2-6/MAGNT.43)