Block Pulse Operational Matrix Method for Solving FRACTIONAL DIFFUSION-WAVE EQUATION

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Abstract:
In this paper, we first introduce two dimensional block pulse functions and the block pulse operational matrices of the fractional order integration. Also, the block pulse operational matrices of the fractional order differentiation are obtained. Then, we consider the computational method based on the above results for solving the fractional DIFFUSION-WAVE EQUATION. The error analysis of the method is given. The method is computationally attractive and applications are demonstrated by some numerical examples.

Keywords: Block pulse functions, Operational matrix, Fractional Diffusion -Wave Equation, Sylvester equation, Error analysis, Numerical solution

1. Introduction

Let $u(x,t)$ be a function, $x \in [0,L]$ and $t \in [0,T]$. Denote by D_t^α the Caputo fractional differential operator at the variable t. Consider the continuous-time fractional diffusion-wave system described by equation

$$D_t^\alpha u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = f(x,t) \qquad (1\text{-}1)$$

with initial and boundary conditions

$$u(x,0) = \varphi(x), u_t(x,0) = 0, \qquad (1\text{-}2)$$

$$u(0,t) = 0, u(L,t) = 0. \qquad (1\text{-}3)$$

Equation (1-1) for $\alpha = 1$ is the classical diffusion equation and for $\alpha = 2$ is the classical wave equation. Thus (1-1) for $\alpha \in (0,2]$ is the diffusion-wave equation. The fractional diffusion-wave equation plays an intermediate role between classical wave and diffusion equations (Weilbeer, 2005), (Jafari & Momani, 2007).

The solution of the homogeneous boundary problem (1-1), (1-2) and (1-3) is given by (Weilbeer, 2005)

$$u(x,t) = \frac{2}{L}\sum_{k=0}^{\infty} c_k E_\alpha\left(-\frac{k^2\pi^2}{L^2} t^\alpha\right) Sin\left(\frac{k\pi}{L} x\right), \qquad (1\text{-}4)$$

Where

$$c_k = \int_0^L \varphi(x) Sin\left(\frac{k\pi}{L} x\right) dx$$

And $E_\alpha(t)$ is the Mittag-Leffler function with one parameter α.

For $\alpha = 1$ and $\alpha = 2$ we obtain respectively

$$E_1\left(-\frac{k^2\pi^2}{L^2} t\right) = \exp\left(-\frac{k^2\pi^2}{L^2} t\right),$$

$$E_2\left(-\frac{k^2\pi^2}{L^2} t^2\right) = Cos\left(\frac{k\pi}{L} t\right)$$

Therefore, from (21) for $\alpha = 1$ and $\alpha = 2$ we obtain the solution of classical diffusion equation and the solution of classical wave equation respectively.

Block pulse functions (BPFs), a set of orthogonal functions with piecewise constant values, have been studied and applied as a useful tool in the synthesis, analysis and other problems of control in recent years. Because of their clearness in expressions and their simplicity in formulations, these functions may have definite advantages for problems involving integrals and derivatives. In this paper, we give numerical method based on block pulse

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operational matrix for solving the fractional DIFFUSION-WAVE EQUATION

The organization of this paper is as follows: In Section 2, some theorems are presented that will be used in later sections. In Section 3, the method is discussed. Section 4 is devoted to numerical experiments and the results are compared with the exact solutions. Section 5 is the conclusion.

2. Preliminaries

In this section, we recall the basic definitions from fractional calculus and some theorems of integral calculus which we shall apply to formulate our new approach.

Definition 2.1. (Podlubny, 1999)A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p\ (> \mu)$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty]$, and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu, m \in \mathbb{N}$.

The Riemann–Liouville fractional integral and Caputo derivative are defined as follows.

Definition 2.2. (Podlubny, 1999)The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau,\ \alpha > 0,\ t > 0. \tag{2-1}$$

Some of the most important properties of operator J^α for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$ are as follows:

1. $J^0 f(t) = f(t)$;
2. $J^\alpha J^\beta f(t) = J^{(\alpha+\beta)} f(t)$;
3. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$;
4. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$;

Definition 2.3. (Podlubny, 1999) The fractional derivative of f (t) in the Caputo sense is defined as

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau,$$

for $m-1 < \alpha \leq m, m \in N,\ t > 0, f \in C_{-1}^m$.

Definition 2.4. (Mohebbi Ghandehari & Ranjbar, 2013)For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,t)}{\partial \tau^m}d\tau, & for\ m-1<\alpha< \\ \frac{\partial^m u(x,t)}{\partial t^m}, & for\ \alpha = m. \end{cases}$$

3. Two dimensional block pulse functions

Block pulse functions of one dimensional have been widely used for differential and integral equations. More details for block pulse functions of one dimensional are given in (Jiang & Schaufelberger, 1992). These conclusions can be extended to the two dimensional block pulse functions.

3.1. Definition and properties

2D-BPFs are defined by

$$\Phi_{i_1,i_2}(x,t) = \begin{cases}1, & (i_1-1)h_1 \leq x \leq i_1 h_1\ and (i_2-1)h_2 \leq t \leq i_2 h_2, \\ 0, & otherwise,\end{cases} \tag{3-1}$$

Where $i_1 = 1,2,\dots,m_1$ and $i_2 = 1,2,\dots,m_2$ with positive integer values for m_1, m_2 and $h_1 = \frac{T_1}{m_1}, h_2 = \frac{T_2}{m_2}, T_1, T_2 \in \mathrm{N}^+$. They have the following properties:

1. Disjointness

$$\Phi_{i_1 i_2}(x,t)\Phi_{j_1 j_2}(x,t) = \begin{cases}\Phi_{i_1,i_2}(x,t), & i_1 = j_1\ and\ i_2 = j_2, \\ 0, & otherwise,\end{cases} \tag{3-2}$$

2. Orthogonality

$$\int_0^{T_1}\int_0^{T_2} \Phi_{i_1 i_2}(x,t)\Phi_{j_1 j_2}(x,t)dtdx = \begin{cases}h_1h_2, & i_1 = j_1\ and\ i_2 = j_2, \\ 0, & otherwise,\end{cases} \tag{3-3}$$

in the region of $x \in [0, T_1)$ and $t \in [0, T_2)$, where $i_1, j_1 = 1,2,\dots,m_1$ and $i_2, j_2 = 1,2,\dots,m_2$.

3. Completeness

For every $u \in L^2([0,T_1) \times [0,T_2))$ when m_1 and m_2 approach to the infinity, Parseval's identity holds:

$$\int_0^{T_1}\int_0^{T_2} u^2(x,t)dtdx = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} u_{i_1,i_2}^2 ||\Phi_{i_1,i_2}(x,t)||^2, \tag{3-4}$$

Where

$$u_{i_1,i_2} = \frac{1}{h_1 h_2}\int_0^{T_1}\int_0^{T_2} \mathrm{u(x,t)}\Phi_{i_1,i_2}(x,t)\,dtdx$$
(3-5)

3.2. BPFs expansion

A function $u(x,t) \in L^2([0,T_1) \times [0,T_2))$ can be expressed as

$$u(x,t) \cong \sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} u_{i_1,i_2}\Phi_{i_1,i_2}(x,t) = V^T\Phi(x,t),$$
(3-6)

where

$V = [u_{1,1}, \dots, u_{1,m_2}, \dots, u_{m_1,1}, \dots, u_{m_1,m_2}]^T$. The block pulse coefficients u_{i_1,i_2} are obtained as

$$u_{i_1,i_2} = \frac{1}{h_1 h_2}\int_{(i_1-1)h_1}^{i_1 h_1}\int_{(i_2-1)h_2}^{i_2 h_2} \mathrm{u(x,t)}\,dtdx$$
(3-7)

Since each two dimensional block pulse function takes only one value in its sub region, the 2D-BPFs can be expanded by the two 1D-BPFs:

$$\Phi_{i_1,i_2}(x,t) = \Phi_{i_1}(x)\Phi_{i_2}(t) \qquad (3\text{-}8)$$

where $\Phi_{i_1}(x)$ and $\Phi_{i_2}(t)$ are the 1D-BPFs related to the variables and t, respectively. Then we have

$$u(x,t) \cong \Phi^T(x)U\Phi(\mathrm{t}) \qquad (3\text{-}9)$$

where

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,m_2} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,m_2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m_1,1} & u_{m_1,2} & \cdots & u_{m_1,m_2} \end{bmatrix},$$
(3-10)

$$\Phi(\mathrm{x}) = [\Phi_1(x), \dots, \Phi_{m_1}(x)]^{\mathrm{T}}, \qquad \Phi(\mathrm{t}) = [\Phi_1(t), \dots, \Phi_{m_2}(t)]^{\mathrm{T}}.$$
(3-11)

3.3. Operational matrix

Now, the operational matrix of fractional integration of block pulse functions was simply introduced. More detailed introduction can be found in (Li & Sun, 2011).

Let $m_1 = m_2 = m$ and $T_1 = T_2 = T$. If J^α is fractional integration operator of block pulse functions, we can get:

$$J^\alpha\Phi(\mathrm{x}) \cong \mathrm{F}_\alpha\Phi(\mathrm{x})$$
(3-12)

where

$$\mathrm{F}_\alpha = \left(\frac{\mathrm{T}}{\mathrm{m}}\right)^\alpha \frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_{m-1} \\ 0 & 1 & \varepsilon_1 & \cdots & \varepsilon_{m-2} \\ 0 & 0 & 1 & \cdots & \varepsilon_{m-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$
(3-13)

And $\varepsilon_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

F_α is called the block pulse operational matrix of fractional integration.

Let D_α is the block pulse operational matrix for the fractional differentiation. According to the property of fractional calculus, $D_\alpha\mathrm{F}_\alpha = \mathrm{I}$, we can obtain easily matrix D_α by inverting the F_α matrix.

4. An approach for solving fractional DIFFUSION-WAVE EQUATION

In this section, we propose our method for finding the numerical solution of fractional DIFFUSION-WAVE EQUATION (1-1).

We suppose $m_1 = m_2 = m$, then we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\Phi^T(x)U\Phi(\mathrm{t})\right) = \left(\frac{\partial^2}{\partial x^2}\Phi^T(x)\right)U\Phi(\mathrm{t}) = \left((\mathrm{D}_2\Phi)^T(x)\right)U\Phi(\mathrm{t}) = \Phi^T(x)\mathrm{D}_2{}^T U\Phi(\mathrm{t})$$
(4-1)

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha}\left(\Phi^T(x)U\Phi(\mathrm{t})\right) = \Phi^T(x)U\frac{\partial^\alpha}{\partial t^\alpha}\Phi(\mathrm{t}) = \Phi^T(x)U\mathrm{D}_\alpha\Phi(\mathrm{t})$$
(4-2)

The function $f(x,t)$ of Eq. (1-1) can be expressed as

$$f(x,t) \cong \Phi^T(x)F\Phi(\mathrm{t}) \qquad (4\text{-}3)$$

Where

$$F = \begin{bmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,m} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1} & f_{m,2} & \cdots & f_{m,m} \end{bmatrix}$$

Substituting Eqs. (4-1)-(4-2) and (4-3) into Eq. (1-1), we have

$$\Phi^T(x)U\mathrm{D}_\alpha\Phi(\mathrm{t}) - \Phi^T(x)\mathrm{D}_2{}^T U\Phi(\mathrm{t}) = \Phi^T(x)F\Phi(\mathrm{t})$$
(4-4)

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Dispersing Eq. (4-4) by the points $(x_{i_1}, t_{i_2}), i_1 = 1,2, \dots, m$ and $i_2 = 1,2, \dots, m$, we can obtain

$$UD_\alpha - D_2{}^T U = F \qquad (4\text{-}5)$$

This is the Sylvester equation.

5. Numerical example

In this section, we solve some examples by our method and compare the numerical results with the exact solutions and some earlier work. To illustrate the accuracy of the method, we compute the error norms L_2 and L_∞ and Maximum error is illustrate the accuracy of the method.

Consider the continuous-time fractional diffusion-wave system described by equation

$$D_t^\alpha u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = Sin(x,t),$$
$$(0 < x, t < 1, \quad 1 \le \alpha \le 2)$$
(5-1)

with initial and boundary conditions

$$u(x,0) = Sin(x), u_t(x,0) = 0,$$
(5-2)

$$u(0,t) = 0, u(1,t) = 0.$$
(5-3)

The numerical results for m = 4, m = 8, m = 16 and m=32 are shown in Figures. 1–4. To obtain numerical solutions of the equation, we need not construct the operational matrix of fractional integration of block pulse functions.

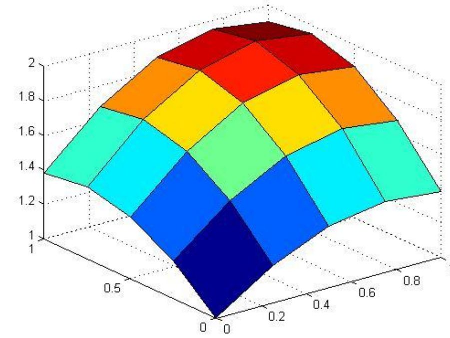

Figure 1. Numerical solution of m = 4.

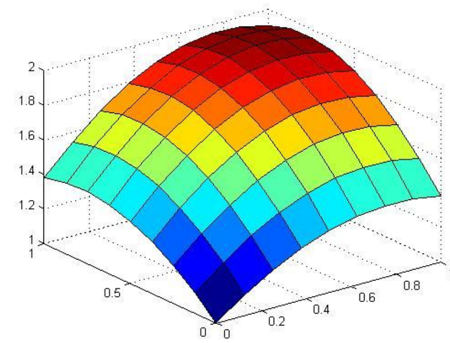

Figure 2.Numerical solution of m = 8.

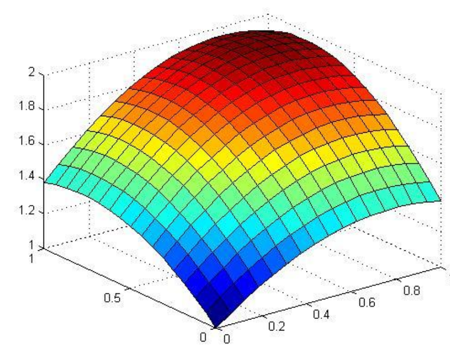

Figure 3. Numerical solution of m = 16.

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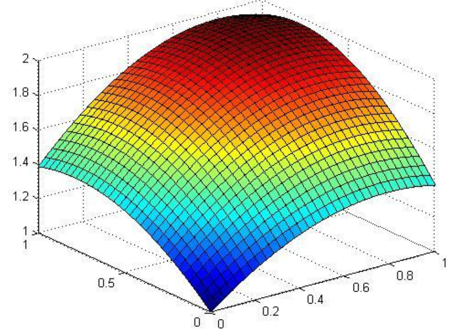

Figure 4. Numerical solution of m = 32.

5. Conclusion

In this article, block pulse operational matrix method was used to solve a class of **FRACTIONAL DIFFUSION-WAVE EQUATION** by combining block pulse functions with operational matrix of fractional differentiation. We translate the initial equation into a Sylvester equation which is easily to solve. Because of the construction of two dimensional block pulse functions is very simple, the advantage of above method can avoid constructing the operational matrix of block pulse functions. One the other hand, since the block pulse operational matrix for the fractional differentiation is upper triangular matrix; it greatly reduces the memory space. What is more, the method in this paper is easy implementation.

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