A novel numerical method for solving system of fractional partial differential equations

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Abstract:
In this paper, we propose a novel method for solving systems of fractional partial differential equations. This is very easy to use method and a combination of the discretization, an interpolation method and nonlinear programming. It can also be applied to equations of other types. The main advantage of the method lies in its flexibility for obtaining the approximate solutions of fractional equations. The fractional derivative is described in the Caputo sense. Using this approach, we convert a system of fractional partial differential equation into a multi objective nonlinear programming problem. Several numerical examples are used to demonstrate the effectiveness and accuracy of the method.

Keywords: System of fractional partial differential equations, Discretization, Nonlinear programming.

1. Introduction

Let $u(x,t)$ be a function, $x \in [0,L]$ and $t \in [0,T]$. Denote by D_t^α the Caputo fractional differential operator at the variable t. Consider the continuous-time fractional diffusion-wave

Consider the system of fractional partial differential equations (FPDEs) with the initial conditions of the form:

$$\sum_{i=1}^{n}\left(\beta_i \frac{\partial^{\alpha_i} u_i(x,t)}{\partial t^{\alpha_i}} + \gamma_i \frac{\partial u_i(x,t)}{\partial x} + \mu_i \frac{\partial^2 u_i(x,t)}{\partial x^2}\right) = f_i(x,t),\ 0 < \alpha_i \le 1,\ i = 1,\dots,n$$

(1-1)

where $t \in \Omega,\ \mathbf{x} \in \Omega', \beta_i, \gamma_i$ *and* $\mu_i, i = 1,2,\dots,n$ *are real parameters with bounded initial conditions* $u_i(x,0) = u_{i0}(x)$ *and boundary conditions* $u_i(0,t) = g_{i1}(t)$ *and* $u_i(1,t) = g_{i2}(t)\ i = 1,2,\dots,n$ *for all* $t \in \Omega$ *and* f_i *are continuous functions.*

This type of fractional differential equations have recently proved to be valuable tools for the modeling of many phenomena in fluid mechanics, physics, electrochemistry, mathematical biology and other sciences (Hilfer, 1999). Various researchers have introduced new methods in the literature. These methods include the Adomian decomposition method (ADM) (Jafari & Seifi, Solving system of nonlinear fractional partial differention equations by homotopy analysis method,, 2009), homotopy analysis method
(HAM) (Jafari & Seifi, Solving system of nonlinear fractional partial differention equations by homotopy analysis method,, 2009), homotopy perturbation method (HPM) (Singh, Gupta, & Rai, 2011) , the variational iteration method (VIM) (Odibat & Momani, Application of variational iteration method to Nonlinear differential equations of fractional order, 2006) and the Laplace decomposition method (Jafari, Khalique, & Nazari, Application of Laplace decomposition method for solving linear and nonlinear fractional diffusion—wave equations, 2011).
In this paper, rather than using these methods, we propose a new numerical approach for solving system of partial

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differential equations of fractional order by using discretization and an interpolation method.
The organization of this paper is as follows: In Section 2, some theorems are presented that will be used in later sections. In Section 3, the method is discussed. Section 4 is devoted to numerical experiments and the results are compared with the exact solutions. Section 5 is the conclusion.

2. Preliminaries

In this section, we recall the basic definitions from fractional calculus and some theorems of integral calculus which we shall apply to formulate our new approach.

Definition 2.1. (Podlubny, 1999)A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in R$, if there exists a real number $p (> \mu)$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty]$, and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu, m \in N$.

The Riemann–Liouville fractional integral and Caputo derivative are defined as follows.

Definition 2.2. (Podlubny, 1999)The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau,$$
$$\alpha > 0, \ t > 0. \qquad (2\text{-}1)$$

Some of the most important properties of operator J^α for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$ are as follows:

1. $J^0 f(t) = f(t)$;
2. $J^\alpha J^\beta f(t) = J^{(\alpha+\beta)} f(t)$;
3. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$;
4. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$;

Definition 2.3. (Podlubny, 1999) The fractional derivative of f (t) in the Caputo sense is defined as

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau,$$

for
$m-1 < \alpha \leq m, m \in N, \ t > 0, f \in C_{-1}^m$
.

Definition 2.4. (Mohebbi Ghandehari & Ranjbar, 2013)For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,t)}{\partial \tau^m} d\tau, & for\ m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & for\ \alpha = m. \end{cases}$$

Now, we state some theorems of calculus and optimization.

Theorem 2.1. (Mohebbi Ghandehari & Ranjbar, 2013) Let f (x, t) be a given function and a, b, c, d are constants, and let $\{t_1, \dots, t_s\}$ and $\{x_1, \dots, x_k\}$ be sets of supporting points in [a, b] and [c, d] respectively, where $a = t_1 < \dots < t_s = b$ and $c = x_1 < \dots < x_k = d$; then

$$\int_a^b \int_c^d f(x,t)dxdt = \lim_{k,s\to\infty} \sum_{i=1}^{k-1} \sum_{j=1}^{s-1} f(\tau_i, \xi_j)\Delta x_i \Delta t_j,$$

where $\Delta x_i = x_{i+1} - x_i$ and $\Delta t_j = t_{j+1} - t_j$, and τ_i, ξ_j are arbitrary points in the intervals $[x_i, x_{i+1}]$ and $[t_j, t_{j+1}]$ respectively.

Remark 2.2. If we choose the same distance between support points, we obtain the following formula:

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$$\int_a^b \int_c^d f(x,t)dxdt = \lim_{k,s\to\infty} hl \sum_{i=1}^{k-1}\sum_{j=1}^{s-1} f(\tau_i,\xi_j),$$

where $l = \frac{b-a}{k}$, $h = \frac{d-c}{m}$ and τ_i, ξ_j are arbitrary points in the intervals $[x_i, x_{i+1}]$ and $[t_j, t_{j+1}]$ respectively.

Theorem 2.3. (Bazara & Shetty, 1979)Let y = f (x) be a convex function on a convex set; then any local minimum of f is a global one.

Theorem 2.4. (Mohebbi Ghandehari & Ranjbar, 2013) Consider n convex functions $f_1, \dots, f_n$; then $g(x,y) = \sum_{i=1}^{n} \alpha_i f_i(x,y)$ is also a convex function for $\alpha_i \geq 0\ (i = 1,\dots,n)$.

3. An approach for solving fractional partial differential equations

In this section, we propose our method for finding the numerical solution of a system of partial differential equation of fractional order of the form (1-1).

Since every interval such as [a, b] can be transformed into [0,1] by a linear transformation, we are choosing $0 \leq x_i \leq 1, i = 1, \dots, n$ and $0 \leq t \leq 1$.

Let

$$E_i(x,t) = \sum_{i=1}^{n}(\boldsymbol{\beta}_i \frac{\partial^{\alpha_i} \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{t}^{\alpha_i}} + \boldsymbol{\gamma}_i \frac{\partial \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial x} + \boldsymbol{\mu}_i \frac{\partial^2 \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial x^2}) - f_i(x,t), \quad i = 1, \dots, n \tag{3-1}$$

$E_i(x,t)$ are functions and depends on the unknown functions $u_i(x,t), i = 1, \dots, n$, so $E_i(x,t)\colon PC([0,1] \times [0,1]) \to R$, where PC means that they are piecewise continuous on the interval [0, 1] × [0, 1].

Let $(\hat{x}, \hat{t})$ be the numerical solution of (1-1), $E_i(x,t), i = 1, \dots, n,$ is the error function for i-th equation. Then the problem of finding the numerical solution of (1-1) converts to an equivalent multi objective optimization problem, as follows:

$$\begin{gathered} Min_{u_{i(x,t)\in PC([0,1]\times[0,1])}} \parallel E_1(x,t) \parallel_1 \\ \vdots \\ Min_{u_{i(x,t)\in PC([0,1]\times[0,1])}} \parallel E_n(x,t) \parallel_1 \end{gathered} \tag{3-2}$$

Since $\parallel E_i(x,t) \parallel_1 = \int_0^1 \int_0^1 |E_1(x,t)|\, dxdt$, the equation (3-2) transform to

$$\begin{gathered} Min_{u_{i(x,t)\in PC([0,1]\times[0,1])}} \int_0^1 \int_0^1 |E_1(x,t)|\, dxdt \\ \vdots \\ Min_{u_{i(x,t)\in PC([0,1]\times[0,1])}} \int_0^1 \int_0^1 |E_n(x,t)|\, dxdt \end{gathered} \tag{3-3}$$

Theorem 3.1. The continuous functions $u_1(x,t), \dots, u_n(x,t)$ are on $[0,1] \times [0,1]$ are a solution for (1-1); if and only if they are the optimal solution of (3-3) with zero objective function.

Proof. Let $u'_1(x,t), \dots, u'_n(x,t)$ are a solution for (1-1), which are continuous on $[0,1] \times [0,1]$; then we have

$$\sum_{i=1}^{n}(\beta_i \frac{\partial^{\alpha_i} u_i(x,t)}{\partial t^{\alpha_i}} + \gamma_i \frac{\partial u_i(x,t)}{\partial x} + \mu_i \frac{\partial^2 u_i(x,t)}{\partial x^2}) - f_i(x,t) = 0, \qquad i = 1, \dots, n$$

And hence,

$$|\sum_{i=1}^{n}(\beta_i \frac{\partial^{\alpha_i} u_i(x,t)}{\partial t^{\alpha_i}} + \gamma_i \frac{\partial u_i(x,t)}{\partial x} + \mu_i \frac{\partial^2 u_i(x,t)}{\partial x^2}) - f_i(x,t)| = 0, \quad i = 1, \dots, n$$

Since $u'_1(x,t), \dots, u'_n(x,t)$ and f are continuous on their domains, by integrating

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both sides of the last equation on $[0,1] \times [0,1]$ we obtain

$$\int_0^1 \cdots \int_0^1 |\sum_{i=1}^{n}\left(\beta_i \frac{\partial^{\alpha_i} u'_i(x,t)}{\partial t^{\alpha_i}} + \gamma_i \frac{\partial u'_i(x,t)}{\partial x_i} + \mu_i \frac{\partial^2 u'_i(x,t)}{\partial {x_i}^2}\right) - f_i(x,t)|dx_1 \dots dx_n dt = 0, i = 1, \dots, n$$

Hence, $\| E_i(x,t) \|_1 = 0, i = 1, \dots, n,$ and this means that $u'_1(x,t), \dots, u'_n(x,t)$ are the optimal solution of (3-3) with zero objective function. For the converse part, let $u'_1(x,t), \dots, u'_n(x,t)$ are be the optimal solution of (3-3) with zero objective function; since,

$$\left|\sum_{i=1}^{n}\left(\beta_i \frac{\partial^{\alpha_i} u'_i(x,t)}{\partial t^{\alpha_i}} + \gamma_i \frac{\partial u'_i(x,t)}{\partial x_i} + \mu_i \frac{\partial^2 u'_i(x,t)}{\partial x_i^2}\right) - f_i(x,t)\right|, i = 1, \dots, n,$$

are absolute functions, by using Lebesgue integral theorems we have:

$$\left|\sum_{i=1}^{n}\left(\beta_i \frac{\partial^{\alpha_i} u'_i(x,t)}{\partial t^{\alpha_i}} + \gamma_i \frac{\partial u'_i(x,t)}{\partial x_i} + \mu_i \frac{\partial^2 u'_i(x,t)}{\partial {x_i}^2}\right) - f_i(x,t)\right| = 0, i = 1, \dots, n$$

And hence,

$$\sum_{i=1}^{n}\left(\boldsymbol{\beta}_i \frac{\partial^{\alpha_i} \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{t}^{\alpha_i}} + \boldsymbol{\gamma}_i \frac{\partial \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial x} + \boldsymbol{\mu}_i \frac{\partial^2 \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial x^2}\right) = f_i(x,t), i = 1, \dots, n$$

Thus $u'_1(x,t), \dots, u'_n(x,t)$ are the solutions of (1-1).

By Theorem 2.4. the objective function of NLP problem (3-3) is a convex function; hence, if zero is a local minimum of (3-3), it is also its global minimum. **Remark 3.2.** The accuracy of the results can be controlled. For example, if we want the total error to be less than a given number ε, we must solve the following multi objective NLP

$$Min_{u_i(x,t) \in PC([0,1] \times [0,1])} \int_0^1 \cdots \int_0^1 |E_1(x,t)| dx_1 \dots dx_n dxt$$

$$\vdots$$

$$Min_{u_i(x,t) \in PC([0,1] \times [0,1])} \int_0^1 \cdots \int_0^1 |E_n(x,t)| dx_1 \dots dx_n dxt$$

$$s.t.$$

$$\| E_i(x,t) \|_1 \leq \varepsilon, i = 1, \dots, n.$$

Now, for solving multi objective NLP problem (3-3) , by applying theorem 2.2 to the double integrals of (3-3) we obtain

$$\int_0^1 \cdots \int_0^1 | \sum_{i=1}^{n}\left(\boldsymbol{\beta}_i \frac{\partial^{\alpha_i} \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{t}^{\alpha_i}} + \boldsymbol{\gamma}_i \frac{\partial \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{x}} + \boldsymbol{\mu}_i \frac{\partial^2 \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{x}^2}\right) - f_i(x,t)| dx_1 \dots dx_n dxt = \lim_{k,s \to \infty} \sum_{l=1}^{k-1} \sum_{j=1}^{s-1} \delta x \delta t | \sum_{i=1}^{n}\left(\boldsymbol{\beta}_i \frac{\partial^{\alpha_i} \boldsymbol{u}_i(\tau_l,\xi_j)}{\partial \boldsymbol{t}^{\alpha_i}} + \boldsymbol{\gamma}_i \frac{\partial \boldsymbol{u}_i(\tau_l,\xi_j)}{\partial \boldsymbol{x}} + \boldsymbol{\mu}_i \frac{\partial^2 \boldsymbol{u}_i(\tau_l,\xi_j)}{\partial \boldsymbol{x}^2}\right) - f_i(\tau_l,\xi_j)|,$$

(3-4)

where τ_l, ξ_j are arbitrary points in the intervals $[x_{l-1}, x_l]$ and $[t_{j-1}, t_j]$ respectively, $t_j = j\delta t, x_l = l\delta x$, $\delta x = \frac{1}{k}$ and $\delta t = \frac{1}{s}$.

If the upper bounds in each interval are choosing, $\tau_l = x_l, \xi_j = t_j$: (3-4) changes to:

$$\| E_i(x,t) \|_1 = \int_0^1 \int_0^1 | \sum_{i=1}^{n}\left(\boldsymbol{\beta}_i \frac{\partial^{\alpha_i} \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{t}^{\alpha_i}} + \boldsymbol{\gamma}_i \frac{\partial \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{x}} + \boldsymbol{\mu}_i \frac{\partial^2 \boldsymbol{u}_i(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{x}^2}\right) - f_i(x,t)| dx\, dt = \lim_{k,s \to \infty} \frac{1}{ks} \sum_{l=1}^{k-1} \sum_{j=1}^{s-1} | \sum_{i=1}^{n}\left(\boldsymbol{\beta}_i \frac{\partial^{\alpha_i} \boldsymbol{u}_i(x_l,t_j)}{\partial \boldsymbol{t}^{\alpha_i}} + \boldsymbol{\gamma}_i \frac{\partial \boldsymbol{u}_i(x_l,t_j)}{\partial \boldsymbol{x}} + \boldsymbol{\mu}_i \frac{\partial^2 \boldsymbol{u}_i(x_l,t_j)}{\partial \boldsymbol{x}^2}\right) - f_i(x_l,t_j)|,$$

(3-5)

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Since, $u_i(x,t), i = 1, \dots, n,$ are unknowns, we cannot calculate their derivatives and hence, we use the approximate equals of theirs as follows:

Consider n points $\{x_1, \dots, x_k\}$ in the bounded domain [0, 1] and the grid points $\{t_1, \dots, t_s\}$ in the time interval [0, 1], where $x_l = l\delta x = \frac{l}{n}$ and $t_j = j\delta t = \frac{j}{s}$, using the Caputo fractional partial derivative of order $\alpha_i, 0 < \alpha_i < 1, i = 1, \dots, n,$ for the time fractional derivative in the Eq. (1-1), we can approximate the time fractional derivative as

$$\frac{\partial^{\alpha_i} u_i(x_l,t_j)}{\partial t^{\alpha_i}} = \frac{1}{\Gamma(1-\alpha_i)} \int_0^{t_j} \frac{\partial u_i(x_l,\tau)}{\partial \tau} (t_j - \tau)^{-\alpha_i} d\tau$$
$$= \frac{1}{\Gamma(1-\alpha_i)} \sum_{r=1}^{j-1} \int_{r\delta t}^{(r+1)\delta t} \frac{\partial u_i(x_l,\tau)}{\partial \tau} (t_j - \tau)^{-\alpha_i} d\tau$$
$$= \frac{1}{\Gamma(1-\alpha_i)} \sum_{r=1}^{j-1} \int_{r\delta t}^{(r+1)\delta t} \frac{u_i(x_l,t_{r+1}) - u_i(x_l,t_r)}{\delta t} (t_j - \tau)^{-\alpha_i} d\tau$$
$$= \frac{1}{\Gamma(1-\alpha_i)} \sum_{r=1}^{j-1} [\frac{u_{i_{l,r+1}} - u_{i_{l,r}}}{\delta t}] \int_{r\delta t}^{(r+1)\delta t} (t_j - \tau)^{-\alpha_i} d\tau$$
$$= \frac{1}{\Gamma(1-\alpha_i)} \sum_{r=1}^{j-1} \left[\frac{u_{i_{l,r+1}} - u_{i_{l,r}}}{\delta t}\right] [\frac{(j-r)^{1-\alpha_i} - (j-r-1)^{1-\alpha_i}}{1-\alpha_i}] \delta t^{1-\alpha_i}$$
$$= \frac{\delta t^{-\alpha_i}}{\Gamma(2-\alpha_i)} \sum_{r=1}^{j-1} [u_{i_{l,r+1}} - u_{i_{l,r}}][(r+1)^{1-\alpha_i} - (r)^{1-\alpha_i}]$$

(3-6)

Since $\delta t = \frac{1}{s}$, (3-6) leads to

$$\frac{\partial^{\alpha_i} u_i(x_l,t_j)}{\partial t^{\alpha_i}} = \frac{s^{\alpha_i}}{\Gamma(2-\alpha_i)} \sum_{r=1}^{j-1} [u_{i_{l,r+1}} - u_{i_{l,r}}][(r+1)^{1-\alpha_i} - (r)^{1-\alpha_i}]$$

(3-7)

On the other hand, the space derivatives in (3-5) will be replaced by the following finite difference approximation:

$$\frac{\partial u_i(x_l,t_j)}{\partial x} \cong \frac{u_{i_{l+1,j}} - u_{i_{l-1,j}}}{2\delta x} = \frac{k}{2}[u_{i_{l+1,j}} - u_{i_{l-1,j}}]$$

$$\frac{\partial^2 u_i(x_l,t_j)}{\partial x^2} \cong \frac{u_{i_{l+1,j}} - 2u_{i_{l,j}} + u_{i_{l-1,j}}}{(\delta x)^2} = k^2[u_{i_{l+1,j}} - 2u_{i_{l,j}} + u_{i_{l-1,j}}]$$

(3-8)

For $l = 1, \dots, k-1$ and $j = 1, \dots, s$

and

Substituting (3-7) and (3-8) in (3-5), we can rewrite (3-5) as follows:

$$\| E_i(x,t) \|_1 = \lim_{k,s\to\infty} \frac{1}{ks} \sum_{l=1}^{k-1} \sum_{j=1}^{s-1} \left| \sum_{i=1}^{n} \left(\beta_i \frac{s^{\alpha_i}}{\Gamma(2-\alpha_i)} \sum_{r=1}^{j-1} [u_{i_{l,r+1}} - u_{i_{l,r}}][(r+1)^{1-\alpha_i} - (r)^{1-\alpha_i}] + \gamma_i \frac{k}{2}\left[u_{i_{l+1,j}} - u_{i_{l-1,j}}\right] + \mu_i k^2 \left[u_{i_{l+1,j}} - 2u_{i_{l,j}} + u_{i_{l-1,j}}\right] \right) - f_i\left(\frac{l}{k}, \frac{j}{s}\right) \right|, \quad i = 1, \dots, n.$$

(3-9)

With discretization of the initial condition and boundary conditions, as constraints on the objective function we obtain

$$u_{i_{l,0}} = u_{i0}\left(\frac{l}{k}\right), \quad u_{i_{0,j}} = g_{1i}\left(\frac{j}{s}\right), \quad u_{i_{k,j}} = g_{2i}\left(\frac{j}{s}\right).$$

We are now dealing with an NLP problem and can use Matlab software systems to find a solution for this problem.

4. Numerical examples

In this section, we solve some examples by our method and compare the numerical results with the exact solutions and some earlier work. To illustrate the accuracy of the method, we compute the error norms L_2 and L_∞ and Maximum error is illustrate the accuracy of the method.

Example 4.1. Consider the system of fractional partial differential equations (FPDEs)

$$\begin{cases} \frac{\partial^{\alpha_1} u}{\partial t^{\alpha_1}} + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} = \frac{2t^{2-\alpha_1}}{\Gamma(\alpha_1)} + 2x, & 0 < \alpha_1, \alpha_2 \le 1, \\ -\frac{\partial^{\alpha_2} v}{\partial t^{\alpha_2}} + 2\frac{\partial u}{\partial x} + 3\frac{\partial v}{\partial x} + 3\frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha_2}}{\Gamma(\alpha_2)} + 10x + 6, & 0 \le x, t \le 1, \end{cases}$$

with the initial conditions

$$u(x,0) = x^2, \qquad v(x,0) = x^2,$$
$$u(0,t) = t^2, \qquad v(0,t) = -t^2,$$
$$u(1,t) = 1 + t^2, \quad v(1,t) = 1 - t^2,$$

The exact solution of this problem is $u(x,t) = x^2 + t^2$

And

$$v(x,t) = x^2 - t^2.$$

Now, we can use our method to solve this equation. First, the fractional partial differential equation is converted into the following optimization problem:

$$\text{Min}\frac{1}{ks}\sum_{l=1}^{k-1}\sum_{j=1}^{s-1}\left|\frac{s^{\alpha_1}}{\Gamma(2-\alpha_1)}\sum_{r=0}^{j-1}[u_{l,r+1}-u_{l,r}][(r+1)^{1-\alpha_1}-(r)^{1-\alpha_1}] + \frac{k}{2}[u_{l+1,j}-u_{l-1,j}] + k^2[u_{l+1,j}-2u_{l,j}+u_{l-1,j}] - k^2[v_{l+1,j}-2v_{l,j}+v_{l-1,j}] - \frac{2(\frac{j}{s})^{2-\alpha_1}}{\Gamma(2-\alpha_1)} - 2\frac{l}{k}\right|$$

$$\text{Min}\frac{1}{ks}\sum_{l=1}^{k-1}\sum_{j=1}^{s-1}\left|\frac{-s^{\alpha_2}}{\Gamma(2-\alpha_2)}\sum_{r=1}^{j-1}[v_{l,r+1}-v_{l,r}][(r+1)^{1-\alpha_2}-(r)^{1-\alpha_2}] + k[u_{l+1,j}-u_{l-1,j}] + \frac{3k}{2}[v_{l+1,j}-v_{l-1,j}] + 3k^2[v_{l+1,j}-2v_{l,j}+v_{l-1,j}] - \frac{2\left(\frac{j}{s}\right)^{2-\alpha_2}}{\Gamma(2-\alpha_2)} - 10\frac{l}{k} - 6\right|$$

$$s.t.$$

$$u\left(\frac{l}{k},0\right) = (\frac{l}{k})^2, l = 1, \dots, k,$$

$$u\left(0,\frac{j}{s}\right) = (\frac{j}{s})^2, j = 1, \dots, s,$$

$$u\left(1,\frac{j}{s}\right) = 1 + (\frac{j}{s})^2, j = 1, \dots, s,$$

$$v\left(\frac{l}{k},0\right) = (\frac{l}{k})^2, l = 1, \dots, k,$$

$$v\left(0,\frac{j}{s}\right) = -(\frac{j}{s})^2, j = 1, \dots, s,$$

$$v\left(1,\frac{j}{s}\right) = 1 - (\frac{j}{s})^2, j = 1, \dots, s,$$

Let $s = k = 10$ and $\alpha_1 = \alpha_2 = 0.5$. The results are displayed in Tables 1, 2 and 3 and Fig. 1, which show that the numerical solutions agree with the exact solution.

Fig.1 The exact solution of example 4.1. (a) u(x,t) (b) v(x,t)

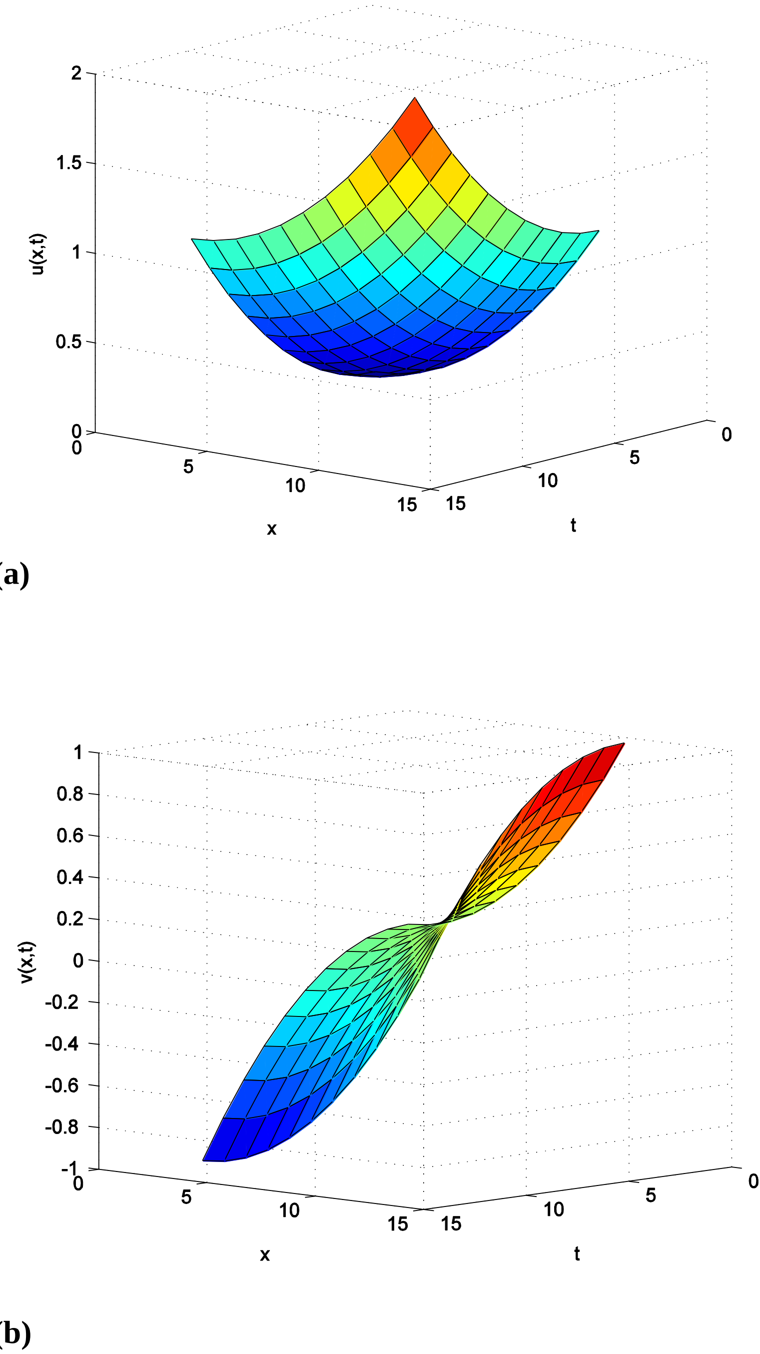

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Table 1. Error norms corresponding to Example 4.1, for $\boldsymbol{\alpha_1 = \alpha_2 = 0.5}$ *and* $\boldsymbol{s = k = 10}$ *in the interval [0, 1].*

Time	**0.1**	**0.5**	**1**
L_∞	1.23421e-003	2.3476e-003	5.7684e-004
L_2	7.3475e-004	7.8603e-004	5.4583e-004

Table 2. Absolute errors of u(x,t) corresponding to Example 4.1, for $\boldsymbol{\alpha_1 = \alpha_2 = 0.5\ s = k = 1}$ *in the interval [0, 1].*

$\boldsymbol{x_i}$	**t=0.1**	**t=0.5**	**t=1**
0.1	3.8765e-004	4.4326e-003	3.4325e-004
0.2	6.9876e-004	5.4565e-003	5.4536e-004
0.3	8.3453e-004	5.4454e-003	6.4563e-004
0.4	1.1254e-003	6.7767e-003	5.6783e-004
0.5	1.3245e-003	7.5468e-004	6.4563e-004
0.6	1.4354e-003	8.5421e-004	6.7682e-004
0.7	2.3456e-003	5.2365e-004	2.3424e-004
0.8	4.5639e-003	3.4682e-004	4.4536e-004
0.9	6.7683e-003	4.3521e-003	5.5543e-003

Table 3. Absolute errors of v(x,t) corresponding to Example 4.1, for $\boldsymbol{\alpha_1 = \alpha_2 = 0.5}$ $\boldsymbol{s = k = 1}$ *in the interval [0, 1].*

$\boldsymbol{x_i}$	**t=0.1**	**t=0.5**	**t=1**
0.1	3.7685e-004	4.4532e-003	3.5123e-004
0.2	5.6754e-004	4.5674e-003	4.1236e-004
0.3	9.3342e-004	6.5676e-003	6.4312e-004
0.4	2.1234e-003	6.6007e-003	5.7384e-004
0.5	1.5436e-003	8.6891e-004	6.5674e-004
0.6	2.9871e-003	9.1231e-004	7.1231e-004
0.7	2.4501e-003	4.3434e-004	3.4325e-004
0.8	3.5643e-003	3.8122e-004	6.6193e-004
0.9	7.2546e-003	4.5324e-003	6.0013e-003

Conclusion

In this paper, we propose a new method for solving systems of fractional partial differential equations. The results show that this scheme is accurate and efficient. In this work, we just need to use some approximate formulas for the derivatives of the unknown functions. By using our method, we reach a discrete problem. Then, we solve a multi objective nonlinear programming problem instead of solving the main fractional partial differential equation.

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