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New Quantum Estimates of Integral Inequalities Via Generalized Preinvex Functions

Muhammad Aslam Noor[a] , Muhammad Uzair Awan[b] , Khalida Inayat Noor[c]

[a,c] COMSATS Institute of Information and Technology, Islamabad, Pakistan
[b] Government College University, Faisalabad, Pakistan

Abstract: In this paper, we establish a new quantum integral identity for twice q -differentiable functions. We use this integral identity as an auxiliary result to derive some new quantum estimates for certain integral inequalities via geenralized preinvex functions. Some special cases are also discussed which can be deduced from our main results. Results obatined in this paper continue to hold for these cases. Ideas and techniques of this paper may be strating point for future research.

Keywords: Convex, invex, preinvex, quantum, differentiable, integral inequalities

1. Introduction

Convexity theory has played an important and fundamental role in the development of various fields of pure and applied sciences. This theory provides us a unified, natural and general framework to study a vide class of unrelated problems. Due to its importance, the concepts of convex sets and convex functions have been extended in different directions. A significant generalization of convex functions is known as preinvex functions, introduced in early 1980's, which inspired many researchers to tackle complicated problems. It has been shown that the minimum of the differentiable preinvex functions can be characterized by a class of variational inequalities, which is called variational-like inequalities. For the applications of the peinvex functions in optimization, see [16,17,18] and the references therein. Noor [19] has shown that a function is a preinvex function if and only if it satisfies Hermite-Hadamard-Noor type of integral inequality. For recent applications and other aspects of the preinvex functions, see [16,17,18]. Gordji et al. [10] introduced the notion of φ-convex functions, which is another type of nonconvex functions.
Also Gorji et al. [10] introduced and investigated another class of convex functions, which includes preinvex functions and φ-convex functions as special cases.

In recent years, much attention has been given to the study of the q-calculus, which has important and significant applications in quantum physics and other branches of sciences. In this paper, we prove a new integral identity for a twice q-differentiable function. This identity is used to derive some new quantum estimates for certain integral inequalities via twice q-differentiable generalized preinvex functions and via its other forms. Several special cases are discussed. An extension of these results for harmonic convex functions is an open problem, which needs further efforts.

Definition 1.1 [10]. *A function $f:\mathcal{K}\to\mathbb{R}$ is said to be φ-convex function, if there exists a bifunction $\varphi(.,.)$, such that*

$$f((1-t)u+tv)\le f(u)+t\varphi(f(v),f(u)), \quad (1.1)$$
$$\forall u,v\in\mathcal{K}, t\in[0,1].$$

We would like to remark that if $\varphi(f(v),f(u))=f(v)-f(u)$, then the φ-convex function becomes convex functions. Clearly every convex function is a φ-convex function, but the converse is not true, see Gordji et al [10].
Also f is said to be φ-affine, if

$$f((1-t)u+tv)=f(u)+t\varphi(f(v),f(u)), \quad \forall u,v\in\mathcal{K}. \quad (1.2)$$

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If $u = v$ in (1.1), then we have $\varphi(f(v), f(u)) \geq 0$. Also, if $t = 1$, then from (1.2), we have $\varphi(f(v), f(u)) = f(v) - f(u)$.

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Definition 1.2. *A function $f : \mathcal{K} \to \mathbb{R}$ is said to be φ-quasiconvex, if*

$$f(tu + (1-t)v) \leq \max\{f(u), f(u) + \varphi(f(y), f(u))\}, \quad \forall u, v \in \mathcal{K}, t \in [0,1].$$

Another significant extenson of the convex functions is known as preinvex function, which was introduced by Ben-Isreal and Mond [2] in 1986. They have shown that the differentiable preinvex functins implies the invex function. Invex functions were introduced by Hansion [11] in mathematical programming. Mohen and Neoghy [15] have shown that the classes of invex functions and preinvex functions are equivalent under suitable conditions. For the sake of completeness and to convey the main ideas involved, we reacll the following concepts.

Definition 1.3 [2]. *A set $\mathcal{K}_\theta \subset \mathbb{R}$ is said to be invex with respect to an arbitrary continuous bifunction $\theta(v,u) : K_\theta \times K_\theta \to \mathbb{R}$, if*

$$u + t\theta(v,u) \in K_\vartheta, \quad \forall u, v \in K_\theta, \ t \in [0,1].$$

Remark 1.4. *Note that if $\theta(v,u) = v - u$ in Definition 1.3, we have definition of classical convex sets. Thus every convex set is invex but the converse is not true, see [2].*

Definition 1.5 [2,36]. *A function $f : \mathcal{K}_\theta \to \mathbb{R}$ is said to be preinvex with respect to an arbitrary continuous bifunction $\theta(.,.) : \mathcal{K}_\theta \times \mathcal{K}_\theta \to \mathbb{R}$, if*

$$f(u + t\theta(v,u)) \leq (1-t) f(u) + t f(v), \quad \forall u, v \in \mathcal{K}_\theta, t \in [0,1].$$

Remark 1.6. *Every convex function is preinvex function with $\theta(v,u) = v - u$ but the converse is not true, see [2, 36].*

Yang et al. [37] introduced the notion of prequasi-invex functions as:

Definition 1.7 [37]. *A function $f : \mathcal{K}_\theta \to \mathbb{R}$ is said to be prequasi-invex with respect to a continuous bifunction $\theta(.,.) : \mathcal{K}_\theta \times \mathcal{K}_\theta \to \mathbb{R}$, if*

$$f(u + t\theta(v,u)) \leq \max\{f(u), f(v)\}, \quad \forall u, v \in \mathcal{K}_\theta, t \in [0,1].$$

For some recent studies on preinvexity and its applications, see [1, 2, 11,15, 17, 18, 19, 21, 24, 25, 29, 30, 31, 32, 36] and the references therein.

We remark that the φ-convex functions and preinvex functions are two different and distinct classes of the convex functions. These two classes of convex functions have different properties and characterizations. Gordji et al. [10] introduced another class of convex functions, which includes these classes of convex functions as special cases. We call it as the generalized preinvex functions.

Definition 1.8 [10]. *Let $\mathcal{K}_\theta \subset \mathbb{R}$ is an invex set with respect to the bifunction $\theta(.,.)$. A function $f : \mathcal{K}_\theta \to \mathbb{R}$ is said to be generalized preinvex with respect to $\theta(.,.)$ and $\varphi(.,.)$, if*

$$f(u + t\theta(v,u)) \leq f(u) + t\varphi(f(v), f(u)), \quad \forall u, v \in \mathcal{K}_\theta, t \in [0,1].$$

For different values of $\theta(.,.)$, and $\varphi(.,.)$, one can easily show that the generalized preinvex functions include φ-convex functions and preinvex functions as special cases.

Definition 1.9. *Let $\mathcal{K}_\theta \subset \mathbb{R}$ be an invex set with respect to the bifunction $\theta(.,.)$. A function $f : \mathcal{K}_\theta \to \mathbb{R}$ is said to be generalized quasi preinvex with respect to $\theta(.,.)$ and $\varphi(.,.)$, if*

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$$f(u+t\theta(v,u)) \leq \max\{f(u), f(u)+\varphi(f(v), f(u))\}, \quad \forall u,v \in \mathcal{K}_\theta, t \in [0,1].$$

A fact which makes theory of convexity more fascinating is its close relationship with theory of inequalities. Many inequalities are proved via convex functions; see [3, 5, 6, 7, 12, 16, 17, 18, 19, 24, 25, 26, 27, 28, 33, 35]. An important inequality which provides the necessary and sufficient condition for convexity of the function is Hermite-Hadamard's inequality. This inequality reads as:

Theorem 1.10 *Let $f: I=[a,b] \to \mathbb{R}$ be a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

The right side of Hermite-Hadamard inequality can be estimated by the inequality of Iyengar, which reads as:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{M(b-a)}{4} - \frac{1}{4M(b-a)}(f(b)-f(a))^2,$$

where by M, we denote the Lipschitz constant,

Gordji et al. [10] derived Hermite-Hadamard type inequality via φ-convex functions as follows:

Theorem 1.11 [10]. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a φ-convex function and φ be bounded from above on $f(I)\times f(I)$. Then, for any $a,b \in I$ with $a<b$, we have*

$$2f\left(\frac{a+b}{2}\right) - M_\varphi \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq f(b) + \frac{\varphi(f(a), f(b))}{2},$$

where M_φ is an upper bound of φ on $f([a,b])\times f([a,b])$.

Noor [17] derived the Hermite-Hadamard inequality via preinvex functions.

Theorem 1.12[17].. *Let $f: \mathcal{K}_\theta \to \mathbb{R}$ be a preinvex function, then*

$$f\left(\frac{2a+\theta(b,a)}{2}\right) \leq \frac{1}{\theta(b,a)}\int_a^{a+\theta(b,a)} f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

Recently several authors have utilized quantum calculus as a strong tool in establishing new extensions of Hermite-Hadamard and other inequalities, see [9, 20, 21, 22, 23, 24, 33, 35]. Having inspiration from the research going on in this direction, we derive some new quantum analogues of Hermite-Hadamard and Iynger type inequalities via twice q-differentiable generalized preinvex and generalized prequasi-invex functions respectively. For this puropose, we derive a new quantum integral identity for twice q-differentiable functions. This auxilliary result plays a crucial role in obtaining the main results of this paper. We also discuss some special cases which are naturally included in our main results. This is the main motivation of this paper.

We now recall some concepts from quantum calculus. To be more precise, Let $J=[a,b] \subseteq \mathbb{R}$ be an interval and $0<q<1$ be a constant. The q-derivative of a function $f: J \to \mathbb{R}$ at a point $x \in J$ on $[a,b]$ is defined as follows.

Definition 1.13 [34,35]. *Let $f: J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f on J at x is defined as*

$$\mathcal{D}_q f(x) = \frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)}, \quad x \neq a.$$

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A function f is q-differentiable on J if $\mathcal{D}_q f(x)$ exists for all $x \in J$.

Definition 1.14 [34,35]. *Let $f : J \to \mathbb{R}$ is a continuous function. A second-order q-derivative on J, which is denoted as $\mathcal{D}_q^2 f$, provided $\mathcal{D}_q f$ is q-differentiable on J is defined as $\mathcal{D}_q^2 f = \mathcal{D}_q(\mathcal{D}_q f) : J \to \mathbb{R}$. Similarly higher order q-derivative on J is defined by $\mathcal{D}_q^n f := J \to \mathbb{R}$.*

Lemma 1.15 [34,35]. *Let $\alpha \in \mathbb{R}$, then*

$$\mathcal{D}_q (x-a)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q}\right)(x-a)^{\alpha-1}.$$

Tariboon et al. [34, 35] defined the q-integral as:

Definition 1.16. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as*

$$\int_a^x f(t)\mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{1.2}$$

for $x \in J$.

These integrals can be viewed as Riemann-type q-integral. Moreover, if $c \in (a,x)$, then the definite q-integral on J is defined by

$$\begin{aligned}\int_c^x f(t)\mathrm{d}_q t &= \int_a^x f(t)\mathrm{d}_q t - \int_a^c f(t)\mathrm{d}_q t \\ &= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &\quad -(1-q)(c-a)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).\end{aligned}$$

Theorem 1.17. *Let $f : I \to \mathbb{R}$ be a continuous function, then*

1. $\mathcal{D}_q \int_a^x f(t)\mathrm{d}_q t = f(x)$
2. $\int_c^x \mathcal{D}_q f(t)\mathrm{d}_q t = f(x) - f(c)$ for $x \in (c,x)$.

Theorem 1.18. *Let $f, g : I \to \mathbb{R}$ be a continuous functions, $\alpha \in \mathbb{R}$, then $x \in J$*

1. $\int_a^x [f(t)+g(t)]\mathrm{d}_q t = \int_a^x f(t)\mathrm{d}_q t + \int_a^x g(t)\mathrm{d}_q t$
2. $\int_a^x (\alpha f(t))(t)\mathrm{d}_q t = \alpha \int_a^x f(t)\mathrm{d}_q t$
3. $\int_a^x f(t)\,{}_a\mathcal{D}_q g(t)\mathrm{d}_q t = (fg)|_c^x - \int_c^x g(qt+(1-q)a)\mathcal{D}_q f(t)\mathrm{d}_q t,$.

for $c \in (a,x)$.

Lemma 1.19. *Let $\alpha \in \mathbb{R}\setminus\{-1\}$, then*

$$\int_a^x (t-a)^{\alpha}\mathrm{d}_q t = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1}.$$

Tariboon et al. [35] obtained the quantum analogue of Hermite-Hadamard's inequality.

Theorem 1.20. *Let $f : I = [a,b] \to \mathbb{R}$ be a convex function, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q x \le \frac{qf(a)+f(b)}{1+q}.$$

Note that when $q \to 1$ in Theorem 1.20, then we have Theorem 1.10.
Noor et al. [21] gave the quantum analogue of Hermite-Hadamard type inequality via preinvex functions.

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Theorem 1.21. *Let $f:\mathcal{K}_\theta \to \mathbb{R}$ be a preinvex function, then*

$$f\left(\frac{2a+\theta(b,a)}{2}\right) \le \frac{1}{\theta(b,a)} \int_a^{a+\theta(b,a)} f(x) \mathrm{d}_q x \le \frac{qf(a)+f(b)}{1+q}.$$

Noor et al. [21] have noticed that when $q \to 1$ in Theorem 1.21, we have Theorem 1.12.
For more details on quantum calculus, see [8, 13, 14].

2 New auxiliary result

In this section, we derive a new quantum integral identity for twice q-differentiable functions.

Lemma 2.1. *Let $f: I_\theta \to \mathbb{R}$ be a twice q-differentiable function on I_θ° such that $\mathcal{D}_q^2 f$ be continuous and integrable on I_θ $0<q<1.$, Then*

$$R_f(a,b;q;\theta) = \frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathcal{D}_q^2 f(a+t\theta(b,a))\mathrm{d}_q t,$$

where

$$R_f(a,b;q;\theta) = \frac{qf(a)+f(a+\theta(b,a))}{1+q} - \frac{1}{\theta(b,a)}\int_a^{a+\theta(b,a)} f(x)\mathrm{d}_q x.$$

Proof. Using Definitions 1.13, 1.14 and 1.16, it suffices to show that

$$= \int_0^1 t(1-qt) \times \left(\frac{qf(a+t\theta(b,a)) - (1+q)f(a+qt\theta(b,a)) + f(a+q^2t\theta(b,a))}{t^2 q(1-q)^2\theta^2(b,a)} \right) \mathrm{d}_q t$$

$$= \left\{ \frac{q\sum_{n=0}^{\infty} f(a+q^n\theta(b,a))}{q(1-q)\theta^2(b,a)} - \frac{(1+q)\sum_{n=0}^{\infty} f(a+q^{n+1}\theta(b,a))}{q(1-q)\theta^2(b,a)} + \frac{\sum_{n=0}^{\infty} f(a+q^{n+2}\theta(b,a))}{q(1-q)\theta^2(b,a)} \right\}$$

$$-q\left\{ \frac{q(1-q)\theta(b,a)\sum_{n=0}^{\infty} q^n f(a+q^n\theta(b,a))}{q(1-q)^2\theta^3(b,a)} \right.$$

$$-\frac{(1+q)(1-q)\theta(b,a)\sum_{n=0}^{\infty} q^{n+1} f(a+q^{n+1}\theta(b,a))}{q^2(1-q)^2\theta^3(b,a)}$$

$$\left. + \frac{(1-q)\theta(b,a)\sum_{n=0}^{\infty} q^{n+2} f(a+q^{n+2}\theta(b,a))}{q^3(1-q)^2\theta^3(b,a)} \right\}$$

$$= \left\{ \frac{q(f(a+\theta(b,a)) - f(a))}{q(1-q)\theta^2(b,a)} - \frac{f(a+q\theta(b,a)) - f(a)}{q(1-q)\theta^2(b,a)} \right\}$$

$$-\left\{\begin{array}{l}\dfrac{1+q}{q^2\theta^3(b,a)}\displaystyle\int_a^{a+\theta(b,a)} f(x)\mathrm{d}_q x\\ +\dfrac{q^2+q-1}{q^2(1-q)\theta^2(b,a)}f(a+\theta(b,a))\\ -\dfrac{f(a+q\theta(b,a))}{q(1-q)\theta^2(b,a)}\end{array}\right\}$$

$$=\frac{qf(a)+f(a+\theta(b,a))}{q^2\theta^2(b,a)}-\frac{1+q}{q^2\theta^3(b,a)}\int_a^b f(a)\mathrm{d}_q x.$$

Multiplying both sides of above equality by $\dfrac{q^2\theta^2(b,a)}{1+q}$, completes the proof. □

Remark 2.2. *We would like to point out that if* $q\to 1$ *in Lemma 2.1, we have Lemma 2.1 [3]. Also if* $q\to 1$ *and* $\theta(b,a)=b-a$ *, we have Lemma 1 [26].*

3 Quantum Hermite-Hadamard inequalities

In this section, using Lemma 2.1, we drive some quantum estimates for Hermite-Hadamard inequalities.

Theorem 3.1. *Let* $f:I_\theta\to\mathbb{R}$ *be a twice* q*-differentiable function on* I_θ° *and let* $\mathcal{D}_q^2 f$ *be continuous and integrable on* I_θ , $0<q<1$.*Let* $|\mathcal{D}_q^2 f(x)|$ *be a generalized preinvex function. Then*

$$R_f(a,b;q;\theta)\le\frac{q^2\theta^2(b,a)}{1+q}\left[\begin{array}{l}|\mathcal{D}_q^2 f(a)|\\ +\zeta(q)\varphi(|\mathcal{D}_q^2 f(b)|,|\mathcal{D}_q^2 f(a)|)\end{array}\right],$$

where

$$\zeta(q)=\frac{1}{(1+q+q^2)(1+q+q^2+q^3)}. \quad (3.1)$$

Proof. Using Lemma 2.1, property of modulus and the fact that $|\mathcal{D}_q^2 f(x)|$ is generalized preinvex function, we have

$$|R_f(a,b;q;\theta)|$$

$$=\left|\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathcal{D}_q^2 f(a+t\theta(b,a))\mathrm{d}_q t\right|$$

$$\le\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)|\mathcal{D}_q^2 f(a+t\theta(b,a))|\mathrm{d}_q t$$

$$\le\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)[|\mathcal{D}_q^2 f(a)|+t\varphi(|\mathcal{D}_q^2 f(b)|,|\mathcal{D}_q^2 f(a)|)]\mathrm{d}_q t$$

$$=\frac{q^2\theta^2(b,a)}{1+q}\left[\begin{array}{l}|\mathcal{D}_q^2 f(a)|\\ +\varphi(|\mathcal{D}_q^2 f(b)|,|\mathcal{D}_q^2 f(a)|)\\ \times\displaystyle\int_0^1 t^2(1-qt)\mathrm{d}_q t\mathrm{d}_q t\end{array}\right]$$

$$=\frac{q^2\theta^2(b,a)}{1+q}\left[\begin{array}{l}|\mathcal{D}_q^2 f(a)|\\ +\left(\dfrac{1}{(1+q+q^2)(1+q+q^2+q^3)}\right)\\ \times\varphi(|\mathcal{D}_q^2 f(b)|,|\mathcal{D}_q^2 f(a)|)\end{array}\right].$$

This completes the proof. □

Theorem 3.2. *Let* $f:I_\theta\to\mathbb{R}$ *be a twice* q*-differentiable function on* I_θ° *and let* $\mathcal{D}_q^2 f$ *be continuous and integrable on* I_θ, $0<q<1$. *If* $|\mathcal{D}_q^2 f(x)|^r$ *is generalized-preinvex function, then, for* $\frac{1}{p}+\frac{1}{r}=1, r>1$ *we have*

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$$R_f(a,b;q;\theta)$$
$$\leq \frac{q^2\theta^2(b,a)}{1+q}\zeta_1^{\frac{1}{p}}(p;q)$$
$$\times\left[\frac{(1+q)|\mathcal{D}_q^2 f(a)|+\varphi(|\mathcal{D}_q^2 f(b)|,|\mathcal{D}_q^2 f(a)|)}{1+q}\right],$$

where

$$\zeta_1(p;q)=(1-q)\sum_{n=0}^{\infty}(q^n)^{p+1}(1-q^{n+1})^p. \quad (3.2)$$

Proof. Using Lemma 2.1, Holder's inequality and the fact that $|\mathcal{D}_q^2 f(x)|^r$ is generalized preinvex function, we have

$$\left|R_f(a,b;q;\theta)\right|$$
$$=\left|\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathcal{D}_q^2 f(a+t\theta(b,a))\mathrm{d}_q t\right|$$
$$\leq \frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)|\mathcal{D}_q^2 f(a+t\theta(b,a))|\mathrm{d}_q t$$
$$\leq \frac{q^2\theta^2(b,a)}{1+q}\left(\int_0^1 (t-qt^2)^p\mathrm{d}_q t\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1[|\mathcal{D}_q^2 f(a)|^r+t\varphi(|\mathcal{D}_q^2 f(b)|^r,|\mathcal{D}_q^2 f(a)|^r)]\mathrm{d}_q t\right)^{\frac{1}{r}}$$
$$=\frac{q^2\theta^2(b,a)}{1+q}\left((1-q)\sum_{n=0}^{\infty}(q^n)^{p+1}(1-q^{n+1})^p\right)^{\frac{1}{p}}$$
$$\left[\frac{(1+q)|\mathcal{D}_q^2 f(a)|+\varphi(|\mathcal{D}_q^2 f(b)|,|\mathcal{D}_q^2 f(a)|^r)}{1+q}\right].$$

This completes the proof. □

Theorem 3.3. *Let* $f: I_\theta\to\mathbb{R}$ *be a twice* q*-differentiable function on* I_θ° *and let t* $\mathcal{D}_q^2 f$ *be continuous and integrable on* I_θ, $0<q<1$. *If* $|\mathcal{D}_q^2 f(x)|^r$ *is generalized preinvex function, then, for* $r\geq 1$, *we have*

$$R_f(a,b;q;\theta)$$
$$\leq \frac{q^2\theta^2(b,a)}{1+q}\zeta_2^{1-\frac{1}{r}}(q)$$
$$\times\left[\begin{array}{l}\zeta_2(q)|\mathcal{D}_q^2 f(a)|^r\\ +\zeta(q)\varphi(|\mathcal{D}_q^2 f(b)|^r,|\mathcal{D}_q^2 f(a)|^r)\end{array}\right]^{\frac{1}{r}},$$

where $\zeta(q)$ is given by (3.1) and

$$\zeta_2(q)=\frac{1}{(1+q)(1+q+q^2)}.$$

Proof. Using Lemma 2.1, power means inequality, and the fact that $|\mathcal{D}_q^2 f(x)|^r$ is generalized preinvex function, we have

$$\left|R_f(a,b;q;\theta)\right|$$
$$=\left|\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathcal{D}_q^2 f(a+t\theta(b,a))\mathrm{d}_q t\right|$$
$$\leq \frac{q^2\theta^2(b,a)}{1+q}\left(\int_0^1 t(1-qt)\mathrm{d}_q t\right)^{1-\frac{1}{r}}$$
$$\left(\int_0^1 t(1-qt)|\mathcal{D}_q^2 f(a+t\theta(b,a))|^r\,\mathrm{d}_q t\right)^{\frac{1}{r}}$$
$$\leq \frac{q^2\theta^2(b,a)}{1+q}\left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}$$
$$\times\left(\begin{array}{l}\int_0^1 t(1-qt)[|\mathcal{D}_q^2 f(a)|^r\\ +t\varphi(|\mathcal{D}_q^2 f(b)|^r,|\mathcal{D}_q^2 f(a)|^r)]\mathrm{d}_q t\end{array}\right)^{\frac{1}{r}}$$
$$=\frac{q^2\theta^2(b,a)}{1+q}\left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}$$
$$\times\left[\left(\frac{q^2}{(1+q)(1+q+q^2)}\right)|\mathcal{D}_q^2 f(a)|^r\right.$$

$$\left.+\left(\frac{1}{(1+q+q^2)(1+q+q^2+q^3)}\right)\times\varphi(|\mathcal{D}_q^2 f(b)|^r,|\mathcal{D}_q^2 f(a)|^r)\right]^{\frac{1}{r}}.$$

This completes the proof. □

4 Quantum Iynger type inequalities

In this section, using Lemma 2.1, we drive some quantum estimates for Iyengar type inequalities via generalized quasi preinvex functions.

Theorem 4.1. *Let* $f:I_\theta\to\mathbb{R}$ *be a twice* q*-differentiable function on* I_θ° *and let t* $\mathcal{D}_q^2 f$ *be continuous and integrable on* I_θ, $0<q<1$. *If* $|\mathcal{D}_q^2 f(x)|$ *is generalize quasi preinvex function, then*

$$R_f(a,b;q;\theta)\le\frac{q^2\theta^2(b,a)}{(1+q)^2(1+q+q^2)}\times\max\{|\mathcal{D}_q^2 f(a)|,|\mathcal{D}_q^2 f(a)|+\varphi(|\mathcal{D}_q^2 f(b)|,|\mathcal{D}_q^2 f(a)|)\}.$$

Proof. Using Lemma 2.1, property of modulus and the fact that $|\mathcal{D}_q^2 f(x)|$ is generalized quasi preinvex function, we have

$$\left|R_f(a,b;q;\theta)\right|$$

$$=\left|\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathcal{D}_q^2 f(a+t\theta(b,a))\mathrm{d}_q t\right|$$

$$\le\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)|\mathcal{D}_q^2 f(a+t\theta(b,a))|\mathrm{d}_q t$$

$$\le\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)\times\max\{|\mathcal{D}_q^2 f(a)|,|\mathcal{D}_q^2 f(a)|+\varphi(|\mathcal{D}_q^2 f(b)|,|\mathcal{D}_q^2 f(a)|)\}\mathrm{d}_q t$$

$$=\frac{q^2\theta^2(b,a)}{(1+q)^2(1+q+q^2)}\times\max\{|\mathcal{D}_q^2 f(a)|,|\mathcal{D}_q^2 f(a)|+\varphi(|\mathcal{D}_q^2 f(b)|,|\mathcal{D}_q^2 f(a)|)\}.$$

This completes the proof. □

Theorem 4.2. *Let* $f:I_\theta\to\mathbb{R}$ *be a twice* q*-differentiable function on* I_θ° *and let* $\mathcal{D}_q^2 f$ *be continuous and integrable on* I_θ, $0<q<1$. *If* $|\mathcal{D}_q^2 f(x)|^r$ *is generalized quasi preinvex function, then, for* $\frac{1}{p}+\frac{1}{r}=1$, $r>1$, *we have*

$$R_f(a,b;q;\theta)\le\frac{q^2\theta^2(b,a)}{1+q}\zeta_1^{\frac{1}{p}}(p;q)\times\left(\begin{array}{l}\max\{|\mathcal{D}_q^2 f(a)|^r,|\mathcal{D}_q^2 f(a)|^r\\+\varphi(|\mathcal{D}_q^2 f(b)|^r,|\mathcal{D}_q^2 f(a)|^r)\}\end{array}\right)^{\frac{1}{r}},$$

where $\zeta_1(p;q)$ is given by (3.2).

Proof. Using Lemma 2.1, Holder's inequality and the fact that $|\mathcal{D}_q^2 f(x)|^r$ is generalized quasi preinvex function, we have

$$\left|R_f(a,b;q;\theta)\right|$$

$$=\left|\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathcal{D}_q^2 f(a+t\theta(b,a))\mathrm{d}_q t\right|$$

$$\le\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)|\mathcal{D}_q^2 f(a+t\theta(b,a))|\mathrm{d}_q t$$

$$\le\frac{q^2\theta^2(b,a)}{1+q}\left(\int_0^1(t-qt^2)^p\mathrm{d}_q t\right)^{\frac{1}{p}}\times\left(\begin{array}{l}\max\{|\mathcal{D}_q^2 f(a)|^r,|\mathcal{D}_q^2 f(a)|^r\\+\varphi(|\mathcal{D}_q^2 f(b)|^r,|\mathcal{D}_q^2 f(a)|^r)\}\end{array}\right)^{\frac{1}{r}}$$

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$$= \frac{q^2\theta^2(b,a)}{1+q}\left((1-q)\sum_{n=0}^{\infty}(q^n)^{p+1}(1-q^{n+1})^p\right)^{\frac{1}{p}}$$
$$\times\begin{pmatrix}\max\{|\mathcal{D}_q^2 f(a)|^r, |\mathcal{D}_q^2 f(a)|^r \\ +\varphi(|\mathcal{D}_q^2 f(b)|^r, |\mathcal{D}_q^2 f(a)|^r)\}\end{pmatrix}^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 4.3. *Let* $f: I_\theta \to \mathbb{R}$ *be a twice* q*-differentiable function on* I_θ° *and let* $\mathcal{D}_q^2 f$ *be continuous and integrable on* I_θ, $0<q<1$. *If* $|\mathcal{D}_q^2 f(x)|^r$ *is generalized quasi preinvex function, then, for* $r \geq 1$, *we have*

$$R_f(a,b;q;\theta)$$
$$\leq \frac{q^2\theta^2(b,a)}{(1+q)^2(1+q+q^2)}$$
$$\times\left[\max\{|\mathcal{D}_q^2 f(a)|^r, \mathcal{D}_q^2 f(b)|^r\}\right]^{\frac{1}{r}}.$$

Proof. Using Lemma 2.1, power means inequality, and the fact that $|\mathcal{D}_q^2 f(x)|^r$ is generalized quasi preinvex function, we have

$$\left|R_f(a,b;q;\theta)\right|$$
$$= \left|\frac{q^2\theta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathcal{D}_q^2 f(a+t\theta(b,a))\,\mathrm{d}_q t\right|$$
$$\leq \frac{q^2\theta^2(b,a)}{1+q}\left(\int_0^1 t(1-qt)\,\mathrm{d}_q t\right)^{1-\frac{1}{r}}$$
$$\times\left(\int_0^1 t(1-qt)|\mathcal{D}_q^2 f(a+t\theta(b,a))|^r\,\mathrm{d}_q t\right)^{\frac{1}{r}}$$
$$\leq \frac{q^2\theta^2(b,a)}{1+q}\left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}$$
$$\times\begin{pmatrix}\int_0^1 t(1-qt)[\max\{|\mathcal{D}_q^2 f(a)|^r, |\mathcal{D}_q^2 f(a)|^r \\ +\varphi(|\mathcal{D}_q^2 f(b)|^r, |\mathcal{D}_q^2 f(a)|^r)\}]\mathrm{d}_q t\end{pmatrix}^{\frac{1}{r}}$$
$$= \frac{q^2\theta^2(b,a)}{(1+q)^2(1+q+q^2)}$$
$$\times\begin{bmatrix}\max\{|\mathcal{D}_q^2 f(a)|^r, |\mathcal{D}_q^2 f(a)|^r \\ +\varphi(|\mathcal{D}_q^2 f(b)|^r, |\mathcal{D}_q^2 f(a)|^r)\}\end{bmatrix}^{\frac{1}{r}}.$$

This completes the proof. □

Conclusion;

In this paper, we have considered and investigated a new class of convex functions in the setting of *q*-calculus, which is called the generalized preinvex functions. We have used the approach of *q*-calculus to derive several new quataum esitmates for the integral inequalities such as Hermite-Hadamard and Iyengar type via twice *q*-diferentaible preinvex functions. All the ressults make significant and important contribuations in this fascinatinag and dynamic field. The interested readers are encouraged to discover new and innovative applications of these quantaum esitamtes for the integral inequalities via generalized prinvex functions in different fields of pure and applied sciences.

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