Hermite-Hadamard Inequalities for Harmonic nonconvex Functions

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Abstract: In this paper, we introduce and consider a new class of harmonic convex functions, which is called harmonic nonconvex (*p*-convex) function. Several new Hermite-Hadamard type inequalities for harmonic nonconvex function are obtained. Some special cases are discussed. Results obtained in this paper continue to hold for these special cases. Our results represent a significant improvement and refinement of the known and new results. The ideas and techniques of this paper may stimulate further research.

Keywords: Harmonic convex functions, Hermite-Hadamard type inequality.

1. Introduction

Inequalities present an attractive and active field of research. In recent years, various inequalities for convex functions and their variant forms are being developed using innovative techniques. Convexity plays an important role in pure and applied mathematics. The concept of convexity has been extended and generalized in several directions to tackle a wide class of problems which arise in pure and applied sciences, see [1, 2, 4, 7, 9, 11, 12, 15, 17, 18, 20] and reference therein. Zhang et al. [20] investigated and studied a new class of convex functions which is called p-convex. For more details, see [13,14].

It is well known that the harmonic mean is the special case of power mean. It is used for the situations when the average of rates is desired and has several applications in trigonometry, geometry, probabilistic and algebra, statistics, physics, computer science, finance and electric circuit theory. The harmonic mean is one of the Pythagorean means, along with the arithmetic and the geometric mean, and is no greater than either of them

Another significant generalization of the convex functions is called the harmonic convex functions, which were investigated by Anderson et al [1] and Iscan [7]. They shown that a function f is a harmonic convex function, if and only if,

$$f\left(\frac{2ab}{a+b}\right)\le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \le \frac{f(a)+f(b)}{2}, \quad x\in[a,b],$$

which is called Hermite-Hadamard inequality for harmonic convex function.

It is clear that the class of p-convex functions and harmonic convex functions are two different classes of convex functions. This fact motivated Noor et al [13] to introduced a new class of convex functions , which is called the harmonic p-convex function. The class of harmonic convex functions is called the harmonic nonconvex functions.

Motivated and inspired by the ongoing research activities in this field, we derive several new Hermite-Hadamard type inequalities are derived for harmonic nonconvex functions. Our results include a wide class of known and new inequalities for convex functions and their variant forms as special cases.

2. Preliminaries

In this section we recall some known concept.

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Definition 2.1. [9]. A set $I=[a,b]\subseteq\mathbb{R}$ is said to be a convex set, if
$(1-t)x+ty\in I,\quad \forall x,y\in I, t\in[0,1].$

Definition 2.2. [9]. A function
$f:I=[a,b]\to\mathbb{R}$ is said to be a convex function, if and only if,
$f((1-t)x+ty)\le(1-t)f(x)+tf(y),$
$\forall x,y\in I, t\in[0,1].$

Definition 2.3. [18]. A set $I=[a,b]\subseteq\mathbb{R}\setminus\{0\}$ is said to be harmonic convex set, if

$$\frac{xy}{tx+(1-t)y}\in I,\quad \forall x,y\in I, t\in[0,1].$$

Definition 2.4. [7]. A function
$f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be harmonic convex function, if and only if,

$$f\left(\frac{xy}{tx+(1-t)y}\right)\le(1-t)f(x)+tf(y),$$

$\forall x,y\in I, t\in[0,1].$

Definition 2.5. [20]. A set $I=[a,b]\subseteq\mathbb{R}$ is said to be nonconvex set if,

$$\left[(1-t)x^p+ty^p\right]^{\frac{1}{p}}\in I,\quad \forall x,y\in I, t\in[0,1],$$

$p\neq 0.$

Definition 2.6. [20]. Let I be a nonconvex set. A function $f:I=[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ is said to be nonconvex function or belongs to the class *PC* (I), if

$$f\left[(1-t)x^p+ty^p\right]^{\frac{1}{p}}\le(1-t)f(x)+tf(y),$$

$\forall x,y\in I, t\in[0,1].$

Theorem 2.1. [13]. Let $f:I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a nonconvex function. If $f\in L[a,b]$, then

$$f\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)\le\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx$$
$$\le\frac{f(a)+f(b)}{2},\ x\in[a,b]. \tag{2.1}$$

Definition 2.7. [17]. Two functions f and g are said to be similarly ordered (f is g-monotone), if and only if,

$$\langle f(x)-f(y),g(x)-g(y)\rangle\ge 0,\qquad \forall x,y\in\mathbb{R}^n.$$

Now we define some new concept.

Definition 2.8. A set $I=[a,b]\subseteq\mathbb{R}\setminus\{0\}$ is said to be harmonic nonconvex set, if

$$\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\in I,\quad \forall x,y\in I, t\in[0,1], p\neq 0.$$

If *p=1*, then Definition 2.8 reduces to Definition 2.3 and if *p=-1*, then Definition 2.8 reduces to Definition 2.1. This shows that the harmonic nonconvex set is more general and include harmonic set and convex set as special cases.

Definition 2.9. A function
$f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be harmonic nonconvex function, if

$$f\left[[\frac{x^py^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right]\le(1-t)f(x)+tf(y),$$

$\forall x,y\in I, t\in[0,1].$

If *p=1*, then Definition 2.9 reduces to Definition 2.4 and if *p=-1*, then Definition 2.9 reduces to Definition 2.2.

Note that for $t=\frac{1}{2}$, we have

$$f\left([\frac{2x^py^p}{x^p+y^p}]^{\frac{1}{p}}\right)\le\frac{f(x)+f(y)}{2},\qquad \forall x,y\in I.$$

which is called Jensen harmonic nonconvex function.

Theorem 3.3 Let f,g be two similarly ordered harmonic nonconvex functions, then the product fg is again a harmonic nonconvex function.

Proof. Let f, g be two harmonic nonconvex functions, Then

$$f\left([\frac{x^py^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)g\left([\frac{x^py^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)$$
$$\le\left[(1-t)f(x)+tf(y)\right]\left[(1-t)g(x)+tg(y)\right]$$
$$=(1-t)^2f(x)g(x)+t(1-t)\begin{bmatrix}f(x)g(y)\\+f(y)g(x)\end{bmatrix}$$

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$$
\begin{aligned}
&+t^2 f(y)g(y)\\
&=(1-t)f(x)g(x)+tf(y)g(y)+(1-t)^2 f(x)g(x)\\
&\quad +t(1-t)\left[f(x)g(y)+f(y)g(x)\right]+t^2 f(y)g(y)\\
&\quad -(1-t)f(x)g(x)-tf(y)g(y)\\
&=(1-t)f(x)g(x)+tf(y)g(y)\\
&\quad -t(1-t)\left[(f(x)-f(y))(g(x)-g(y))\right]\\
&\leq (1-t)f(x)g(x)+tf(y)g(y), \qquad (3.8)
\end{aligned}
$$

This shows that the product of two harmonic nonconvex functions is harmonic nonconvex function.

We need the following simple but important fact.

Remark 2.1. Let $I=[a,b]\subseteq \mathbb{R}\setminus\{0\}$. If we consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$ defined by $g(t)=f\left(\frac{1}{t}\right)$, then f is harmonic nonconvex on $[a,b]$, if and only if, g is nonconvex in the usual sense on $\left[\frac{1}{b},\frac{1}{a}\right]$.

3. Main Results

We prove the following result using essentially the technique of Dragomir [4].

Theorem 3.1. Let $g:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ and $\lambda\in[0,1]$, then

$$
\begin{aligned}
&\int_0^1 g\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\mathrm{d}t=(1-\lambda)\\
&\times\int_0^1 g\left[(1-t)\left[(1-\lambda)a^p+\lambda b^p\right]+tb^p\right]^{\frac{1}{p}}\mathrm{d}t\\
&+\lambda\int_0^1 g\left[(1-t)a^p+t\left[(1-\lambda)a^p+\lambda b^p\right]\right]^{\frac{1}{p}}\mathrm{d}t.
\end{aligned}
\qquad (3.1)
$$

Proof. For $\lambda=0$ and $\lambda=1$, the inequality (3.1) is obvious. Let $\lambda\in(0,1)$. Observe that

$$
\begin{aligned}
&\int_0^1 g\left[(1-t)\left[(1-\lambda)a^p+\lambda b^p\right]+tb^p\right]^{\frac{1}{p}}\mathrm{d}t\\
&=\int_0^1 g\left[\left[(1-t)\lambda+t\right]b^p+(1-t)(1-\lambda)a^p\right]^{\frac{1}{p}}\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 g\left[(1-t)a^p+t\left[(1-\lambda)a^p+\lambda b^p\right]\right]^{\frac{1}{p}}\mathrm{d}t\\
&=\int_0^1 g\left[t\lambda b^p+(1-\lambda t)a^p\right]^{\frac{1}{p}}\mathrm{d}t
\end{aligned}
$$

If we make the change of variable $u:=(1-t)\lambda+t$ then we have

$$
\begin{aligned}
&\int_0^1 g\left[\left[(1-t)\lambda+t\right]b^p+(1-t)(1-\lambda)a^p\right]^{\frac{1}{p}}\mathrm{d}t\\
&=\frac{1}{1-\lambda}\int_\lambda^1 g\left[ub^p+(1-u)a^p\right]^{\frac{1}{p}}\mathrm{d}u
\end{aligned}
$$

If we make the change of variable $u:=\lambda t$ then we have

$$
\begin{aligned}
&\int_0^1 g\left[t\lambda b^p+(1-\lambda t)a^p\right]^{\frac{1}{p}}\mathrm{d}t\\
&=\frac{1}{\lambda}\int_0^\lambda g\left[ub^p+(1-u)b^p\right]^{\frac{1}{p}}\mathrm{d}u
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_0^1 g\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\mathrm{d}t=(1-\lambda)\\
&\quad\times\int_0^1 g\left[(1-t)\left[(1-\lambda)a^p+\lambda b^p\right]+tb^p\right]^{\frac{1}{p}}\mathrm{d}t\\
&\quad+\lambda\int_0^1 g\left[(1-t)a^p+t\left[(1-\lambda)a^p+\lambda b^p\right]\right]^{\frac{1}{p}}\mathrm{d}t\\
&=\int_\lambda^1 g\left[ub^p+(1-u)a^p\right]^{\frac{1}{p}}\mathrm{d}u\\
&\quad+\int_0^\lambda g\left[ub^p+(1-u)a^p\right]^{\frac{1}{p}}\mathrm{d}u\\
&=\int_0^1 g\left[ub^p+(1-u)a^p\right]^{\frac{1}{p}}\mathrm{d}u
\end{aligned}
$$

and the identity (3.1) is proved.

I). If $p=1$, then Theorem 3.1 reduces to the following result.

Corollary 3.1. [4]. Let $g:I=[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ and $\lambda\in[0,1]$, then

$$
\begin{aligned}
&\int_0^1 g\left[(1-t)a+tb\right]\mathrm{d}t\\
&=(1-\lambda)\int_0^1 g\left[(1-t)\left[(1-\lambda)a+\lambda b\right]+tb\right]\mathrm{d}t\\
&\quad+\lambda\int_0^1 g\left[(1-t)a+t\left[(1-\lambda)a+\lambda b\right]\right]\mathrm{d}t
\end{aligned}
$$

II). If $p=-1$, then Theorem 3.1 reduces to the following result.

Corollary 3.2. [4]. Let $g: I=[a,b]\subseteq \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ and $\lambda\in[0,1]$, then

$$\int_0^1 g\left[\frac{ab}{(1-t)b+ta}\right]\mathrm{d}t$$

$$=(1-\lambda)\int_0^1 g\left[\frac{ab}{(1-t)[(1-\lambda)b+\lambda a]+ta}\right]\mathrm{d}t$$

$$+\lambda\int_0^1 g\left[\frac{ab}{(1-t)b+t[(1-\lambda)b+\lambda a]}\right]\mathrm{d}t.$$

Theorem 3.2. Let $f: I=[a,b]\subseteq \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a harmonic nonconvex function on the interval $[a,b]$.Then for any $\lambda\in[0,1]$, we have

$$f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)$$

$$\le(1-\lambda)f\left([\frac{2a^pb^p}{(1-\lambda)a^p+(\lambda+1)b^p}]^{\frac{1}{p}}\right)$$

$$+\lambda f\left([\frac{2a^pb^p}{(2-\lambda)a^p+\lambda b^p}]^{\frac{1}{p}}\right)$$

$$\le\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}\mathrm{d}x$$

$$\le\frac{1}{2}\left[\begin{array}{l} f\left([\frac{a^pb^p}{(1-\lambda)a^p+\lambda b^p}]^{\frac{1}{p}}\right)+(1-\lambda)f(a) \\ +\lambda f(b)\end{array}\right]$$

$$\le\frac{f(a)+f(b)}{2}. \qquad (3.2)$$

Proof. Let g is nonconvex function on $\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=f\left(\frac{1}{s}\right)$, $s\in\left[\frac{1}{b},\frac{1}{a}\right]$, then by Hermite Hadamard inequality for nonconvex functions (2.1), we have for $\lambda\in[0,1]$

$$g\left([\frac{(1-\lambda)a^p+(\lambda+1)b^p}{2a^pb^p}]^{\frac{1}{p}}\right)$$

$$=g\left([\frac{(1-\lambda)\frac{1}{b^p}+\lambda\frac{1}{a^p}+\frac{1}{a^p}}{2}]^{\frac{1}{p}}\right)$$

$$\le\int_0^1 g\left(\left[(1-t)\left((1-\lambda)\frac{1}{b^p}+\lambda\frac{1}{a^p}\right)+t\frac{1}{a^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t$$

$$\le\frac{g\left(\left[(1-\lambda)\frac{1}{b^p}+\lambda\frac{1}{a^p}\right]^{\frac{1}{p}}\right)+g\left(\frac{1}{a}\right)}{2}$$

$$=\frac{g\left([\frac{(1-\lambda)a^p+\lambda b^p}{a^pb^p}]^{\frac{1}{p}}\right)+g\left(\frac{1}{a}\right)}{2} \qquad (3.3)$$

and

$$g\left([\frac{(2-\lambda)a^p+\lambda b^p}{2a^pb^p}]^{\frac{1}{p}}\right)$$

$$=g\left([\frac{\frac{1}{b^p}+(1-\lambda)\frac{1}{b^p}+\lambda\frac{1}{a^p}}{2}]^{\frac{1}{p}}\right)$$

$$\le\int_0^1 g\left(\left[(1-t)\frac{1}{b^p}+t\left((1-\lambda)\frac{1}{b^p}+\lambda\frac{1}{a^p}\right)\right]^{\frac{1}{p}}\right)\mathrm{d}t$$

$$\le\frac{g\left(\frac{1}{b}\right)+g\left(\left[(1-\lambda)\frac{1}{b^p}+\lambda\frac{1}{a^p}\right]^{\frac{1}{p}}\right)}{2}$$

$$=\frac{g\left(\frac{1}{b}\right)+g\left([\frac{(1-\lambda)a^p+\lambda b^p}{a^pb^p}]^{\frac{1}{p}}\right)}{2} \qquad (3.4)$$

If we multiply $(1-\lambda)$ by (3.3) and λ by (3.4), adding the resultant, we have

$$(1-\lambda)f\left([\frac{2a^pb^p}{(1-\lambda)a^p+(\lambda+1)b^p}]^{\frac{1}{p}}\right)$$
$$+\lambda f\left([\frac{2a^pb^p}{(2-\lambda)a^p+\lambda b^p}]^{\frac{1}{p}}\right)$$
$$\le\int_0^1 f\left([\frac{a^pb^p}{(1-t)a^p+tb^p}]^{\frac{1}{p}}\right)\mathrm{d}t$$
$$\le(1-\lambda)\frac{f\left([\frac{a^pb^p}{(1-\lambda)a^p+\lambda b^p}]^{\frac{1}{p}}\right)+f(a)}{2}$$
$$+\lambda\frac{f(b)+f\left([\frac{a^pb^p}{(1-\lambda)a^p+\lambda b^p}]^{\frac{1}{p}}\right)}{2}$$
$$=\frac{1}{2}\left[\begin{array}{l} f\left([\frac{a^pb^p}{(1-\lambda)a^p+\lambda b^p}]^{\frac{1}{p}}\right)\\ +(1-\lambda)f(a)+\lambda f(b)\end{array}\right] \quad (3.5)$$

Also

$$(1-\lambda)[f\left([\frac{2a^pb^p}{(1-\lambda)a^p+(\lambda+1)b^p}]^{\frac{1}{p}}\right)$$
$$+\lambda f\left([\frac{2a^pb^p}{(2-\lambda)a^p+\lambda b^p}]^{\frac{1}{p}}\right)$$
$$=(1-\lambda)g\left([\frac{(1-\lambda)a^p+(\lambda+1)b^p}{2a^pb^p}]^{\frac{1}{p}}\right)$$
$$+\lambda g\left([\frac{(2-\lambda)a^p+\lambda b^p}{2a^pb^p}]^{\frac{1}{p}}\right)$$
$$\ge g\left(\left[\begin{array}{l}\frac{(1-\lambda)[(1-\lambda)a^p+(\lambda+1)b^p]}{2a^pb^p}\\ +\frac{\lambda[(2-\lambda)a^p+\lambda b^p]}{2a^pb^p}\end{array}\right]^{\frac{1}{p}}\right)$$
$$=g\left([\frac{a^p+b^p}{2a^pb^p}]^{\frac{1}{p}}\right)=f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right) \quad (3.6)$$

and

$$f\left([\frac{a^pb^p}{(1-\lambda)a^p+\lambda b^p}]^{\frac{1}{p}}\right)+(1-\lambda)f(a)+\lambda f(b)$$
$$\le(1-\lambda)f(b)+\lambda f(a)+(1-\lambda)f(a)+\lambda f(b)$$
$$=f(a)+f(b), \quad (3.7)$$

From (3.5)-(3.7), we obtained the desired inequality (3.2).

Some special cases of Theorem (3.2) are as follows:

I). If $p=1$, then we have

Corollary 3.3. [4]. Let $f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a harmonic convex function on the interval $[a,b]$.Then for any $\lambda\in[0,1]$, we have

$$f\left(\frac{2ab}{a+b}\right)\le(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)$$
$$+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)$$
$$\le\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x$$
$$\le\frac{1}{2}\left[\begin{array}{l} f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)\\ +(1-\lambda)f(a)+\lambda f(b)\end{array}\right]$$
$$\le\frac{f(a)+f(b)}{2}.$$

II). If $p=1$ and $\lambda=\frac{1}{2}$, then we have

Corollary 3.4. [4].. Let $f:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a harmonic convex function, then

$$f\left(\frac{2ab}{a+b}\right)\le\frac{1}{2}\left[f\left(\frac{4ab}{a+3b}\right)+f\left(\frac{4ab}{3a+b}\right)\right]$$
$$\le\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x$$
$$\le\frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right)+\frac{f(a)+f(b)}{2}\right]$$
$$\le\frac{1}{2}[f(a)+f(b)].$$

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III). If $p=-1$ and $\lambda=\frac{1}{2}$, then we have

Corollary 3.5. [9]. Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]$$
$$\le\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$
$$\le\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]$$
$$\le\frac{1}{2}[f(a)+f(b)].$$

Theorem 3.4 . Let $f,g: I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic nonconvex functions. If $fg\in L[a,b]$, then

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x$$
$$\le\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$
$$\le\frac{1}{6}\left\{\begin{matrix}[M_1(a,b)]^2+[N_1(a,b)]^2\\ -[f(a)f(b)+g(a)g(b)]\end{matrix}\right\},$$
$$\le\frac{1}{12}\left\{\begin{matrix}[f(a)+g(a)]^2+[f(b)+g(b)]^2\\ +[f(a)+g(a)][f(b)+g(b)]\end{matrix}\right\},$$

where

$$M_1=f(a)+f(b) \tag{3.9}$$
$$N_1=g(a)+g(b) \tag{3.10}$$
$$M(a,b)=f(a)g(a)+f(b)g(b) \tag{3.11}$$
$$N(a,b)=f(a)g(b)+f(b)g(a). \tag{3.12}$$

Proof.

Let f,g be harmonic nonconvex functions. Then

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x$$
$$=\int_0^1 f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t$$
$$\le\int_0^1\left[(1-t)f(a)+tf(b)\right]\left[(1-t)g(a)+tg(b)\right]\mathrm{d}t$$
$$=f(a)g(a)\int_0^1(1-t)^2\mathrm{d}t+\begin{bmatrix}f(a)g(b)\\ +f(b)g(a)\end{bmatrix}\int_0^1 t(1-t)\mathrm{d}t$$
$$+f(b)g(b)\int_0^1 t^2\mathrm{d}t$$
$$=\frac{1}{3}\begin{bmatrix}f(a)g(a)\\ +f(b)g(b)\end{bmatrix}+\frac{1}{6}\begin{bmatrix}f(a)g(b)\\ +f(b)g(a)\end{bmatrix}$$
$$=\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)$$
$$\le\int_0^1\left\{\frac{[(1-t)f(a)+tf(b)]^2+[(1-t)g(a)+tg(b)]^2}{2}\right\}\mathrm{d}t$$
$$=\frac{1}{2}\int_0^1\left\{\begin{matrix}(1-t)^2\left[(f(a))^2+(g(a))^2\right]\\ +t^2\left[(f(b))^2+(g(b))^2\right]\\ +2t(1-t)\left[(f(a)f(b)+g(a)g(b)\right]\end{matrix}\right\}\mathrm{d}t$$
$$=\frac{1}{6}\left\{\begin{matrix}[f(a)+f(b)]^2+[g(a)+g(b)]^2\\ -[f(a)f(b)+g(a)g(b)]\end{matrix}\right\}$$
$$=\frac{1}{6}\left\{\begin{matrix}[M_1(a,b)]^2+[N_1(a,b)]^2\\ -[f(a)f(b)+g(a)g(b)]\end{matrix}\right\}$$
$$\le\int_0^1\left\{\frac{[(1-t)f(a)+tf(b)+(1-t)g(a)+tg(b)]^2}{4}\right\}\mathrm{d}t$$
$$=\frac{1}{4}\int_0^1\left\{\begin{matrix}(1-t)^2[f(a)+g(a)]^2+t^2[f(b)+g(b)]^2\\ +2t(1-t)[f(a)+g(a)][f(b)+g(b)]\end{matrix}\right\}\mathrm{d}t$$
$$=\frac{1}{12}\left\{\begin{matrix}[f(a)+g(a)]^2+[f(b)+g(b)]^2\\ +[f(a)+g(a)][f(b)+g(b)]\end{matrix}\right\}$$

If $p=1$, then Theorem 3.4 reduces to the following result.

Corollary 3.6

Let $f,g: I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic convex functions. If $fg\in L[a,b]$, then

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$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x$$
$$\leq \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$
$$\leq \frac{1}{6}\left\{\begin{matrix}[M_1(a,b)]^2+[N_1(a,b)]^2\\ -[f(a)f(b)+g(a)g(b)]\end{matrix}\right\},$$
$$\leq \frac{1}{12}\left\{\begin{matrix}[f(a)+g(a)]^2+[f(b)+g(b)]^2\\ +[f(a)+g(a)][f(b)+g(b)]\end{matrix}\right\},$$
where $M_1(a,b)$, $N_1(a,b)$, $M(a,b)$ and $N(a,b)$ are given by (3.9) - (3.12).

If $p=-1$, then Theorem 3.4 reduces to the following result.

Corollary 3.7. [11]. Let $f,g: I\subset\mathbb{R}\to\mathbb{R}$ be convex functions. If $fg\in L[a,b]$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x$$
$$\leq \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$
$$\leq \frac{1}{6}\left\{\begin{matrix}[M_1(a,b)]^2+[N_1(a,b)]^2\\ -[f(a)f(b)+g(a)g(b)]\end{matrix}\right\},$$
$$\leq \frac{1}{12}\left\{\begin{matrix}[f(a)+g(a)]^2+[f(b)+g(b)]^2\\ +[f(a)+g(a)][f(b)+g(b)]\end{matrix}\right\},$$
where $M_1(a,b)$, $N_1(a,b)$, $M(a,b)$ and $N(a,b)$ are given by (3.9) - (3.12).

Theorem 3.5

Let $f,g: I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be similarly ordered harmonic nonconvex functions, then

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x \leq \frac{1}{2}M(a,b),$$
where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Proof.

Integrating inequality (3.8) completes the proof.

Theorem 3.6

Let $f,g: I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic nonconvex functions. If $fg\in L[a,b]$, then

$$2f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)$$
$$-\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x$$
$$\leq \frac{1}{6}M(a,b)+\frac{1}{3}N(a,b).$$
where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Proof. Let f,g be two harmonic nonconvex functions, with $t=\frac{1}{2}$. Then

$$f\left([\frac{2x^py^p}{x^p+y^p}]^{\frac{1}{p}}\right)\leq \frac{f(x)+f(y)}{2}, \qquad \forall x,y\in I,$$

$$g\left([\frac{2x^py^p}{x^p+y^p}]^{\frac{1}{p}}\right)\leq \frac{g(x)+g(y)}{2}, \qquad \forall x,y\in I.$$

Let $x=\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}$, and

$y=\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}$. Thus

$$f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\leq \frac{1}{2}\left[\begin{matrix}f\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)\\ +f\left([\frac{a^pb^p}{(1-t)a^p+tb^p}]^{\frac{1}{p}}\right)\end{matrix}\right]$$

$$g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\leq \frac{1}{2}\left[\begin{matrix}g\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)\\ +g\left([\frac{a^pb^p}{(1-t)a^p+tb^p}]^{\frac{1}{p}}\right)\end{matrix}\right].$$

Consider

$$f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)$$

$$\leq\frac{1}{4}\left\{\begin{array}{l}\left[f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\right]\\ \left[g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)+g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\right]\end{array}\right\}$$

$$=\frac{1}{4}\left[\begin{array}{l}f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\\ +f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\\ +f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\\ +f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\end{array}\right]$$

$$\leq\frac{1}{4}\left\{\begin{array}{l}f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\\ +f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\\ +[(1-t)f(a)+tf(b)][tg(a)+(1-t)g(b)]\\ +[tf(a)+(1-t)f(b)][(1-t)g(a)+tg(b)]\end{array}\right\}$$

$$=\frac{1}{4}\left\{\begin{array}{l}\int_0^1 f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\ +\int_0^1 f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\ +2[f(a)g(a)+f(b)g(b)]\int_0^1 t(1-t)\mathrm{d}t\\ +[f(a)g(b)+f(b)g(a)]\int_0^1(t^2+(1-t)^2)\mathrm{d}t\end{array}\right\}$$

$$=\frac{1}{4}\left\{\begin{array}{l}\int_0^1 f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\\ g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\ +\int_0^1 f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\\ g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\ +\frac{1}{3}[f(a)g(a)+f(b)g(b)]\\ +\frac{2}{3}[f(a)g(b)+f(b)g(a)]\end{array}\right\}$$

$$=\frac{1}{2}\left[\begin{array}{l}\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x\\ +\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)\end{array}\right],$$

which is the required result.

If $p=1$, then Theorem 3.6 reduces to the following result.

Corollary 3.8. Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic convex functions. If $fg\in L[a,b]$, then

$$2f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)g(x)}{x^2}\mathrm{d}x$$

$$\leq\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b).$$

where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

If $p=-1$, then Theorem 3.6 reduces to the following result.

Corollary 3.9. [16]. Let $f,g:I\subset\mathbb{R}\to\mathbb{R}$ be convex functions. If $fg\in L[a,b]$, then

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x$$

$$\leq\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b).$$

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where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Theorem 3.7. Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic nonconvex functions. If $fg\in L[a,b]$, then

$$\int_a^b\int_a^b\int_0^1 f\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right) g\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)\mathrm{d}t\mathrm{d}y\mathrm{d}x$$
$$\leq \frac{2(b^p-a^p)}{3}\int_a^b f(x)g(x)\mathrm{d}x + \frac{(b^p-a^p)^2x^{(p+1)^2}}{12p^2a^{2p}b^{2p}}\left[M(a,b)+N(a,b)\right],$$

where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Proof.

Let *f,g* be two harmonic nonconvex functions on *I*. Then

$$f\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)\leq(1-t)f(x)+tf(y),$$
$$g\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)\leq(1-t)g(x)+tg(y),$$

$\forall x,y\in I, t\in[0,1]$.

Thus

$$f\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)g\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)$$
$$\leq\left[(1-t)f(x)+tf(y)\right]\left[(1-t)g(x)+tg(y)\right]$$
$$=(1-t)^2f(x)g(x)+t^2f(y)g(y) + t(1-t)\left[f(x)g(y)+g(x)f(y)\right]$$

Integrating over $[0,\ 1]$, we have

$$\int_0^1 f\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)g\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)\mathrm{d}t$$
$$\leq f(x)g(x)\int_0^1(1-t)^2\mathrm{d}t+f(y)g(y)\int_0^1 t^2\mathrm{d}t + \left[f(x)g(y)+g(x)f(y)\right]\int_0^1 t(1-t)\mathrm{d}t$$
$$=\frac{1}{3}\left[f(x)g(x)+f(y)g(y)\right] + \frac{1}{6}\left[f(x)g(y)+g(x)f(y)\right]$$

Now, integrating both sides on $[a,b]\times[a,b]$, we have

$$\int_a^b\int_a^b\int_0^1 f\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right) g\left([\frac{x^p y^p}{tx^p+(1-t)y^p}]^{\frac{1}{p}}\right)\mathrm{d}t\mathrm{d}y\mathrm{d}x$$
$$\leq\frac{2(b-a)}{3}\int_a^b f(x)g(x)\mathrm{d}x+\frac{1}{6}\int_a^b g(y)\mathrm{d}y\int_a^b f(x)\mathrm{d}x + \frac{1}{6}\int_a^b f(y)\mathrm{d}y\int_a^b g(x)\mathrm{d}x$$

$$\leq\frac{2(b-a)}{3}\int_a^b f(x)g(x)\mathrm{d}x + \frac{(b^p-a^p)^2x^{(p+1)^2}}{6p^2a^{2p}b^{2p}}\left[\begin{array}{l}\left(\frac{g(a)+g(b)}{2}\right)\left(\frac{f(a)+f(b)}{2}\right)\\ +\left(\frac{f(a)+f(b)}{2}\right)\left(\frac{g(a)+g(b)}{2}\right)\end{array}\right]$$

$$=\frac{2(b-a)}{3}\int_a^b f(x)g(x)\mathrm{d}x + \frac{(b^p-a^p)^2x^{(p+1)^2}}{12p^2a^{2p}b^{2p}}\left[(f(a)+f(b)(g(a)+g(b)\right]$$

$$=\frac{2(b-a)}{3}\int_a^b f(x)g(x)\mathrm{d}x + \frac{(b^p-a^p)^2x^{(p+1)^2}}{12p^2a^{2p}b^{2p}}\left[M(a,b)+N(a,b)\right],$$

which is the required result.

If *p=1*, then Theorem 3.7 reduces to the following result which appears to be new one.

Corollary 3.10. Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic convex functions. If $fg\in L[a,b]$, then

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$$\frac{3a^2b^2}{2(b-a)^2}\int_a^b\int_a^b\int_0^1 f\left(\frac{xy}{tx+(1-t)y}\right)g\left(\frac{xy}{tx+(1-t)y}\right)\mathrm{d}t\mathrm{d}y\mathrm{d}x$$
$$\le\frac{a^2b^2}{b-a}\int_a^b f(x)g(x)\mathrm{d}x+\frac{x^4}{8}[M(a,b)+N(a,b)],$$
where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

If $p=-1$, then Theorem 3.7 reduces to the following result.

Corollary 3.11 [3]

Let $f,g:I\subset\mathbb{R}\to\mathbb{R}$ be convex functions. If $fg\in L[a,b]$, then
$$\frac{3}{2(b-a)^2}\int_a^b\int_a^b\int_0^1 f\left((1-t)x+ty\right)g\left((1-t)x+ty\right)\mathrm{d}t\mathrm{d}y\mathrm{d}x$$
$$\le\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x+\frac{1}{8}[M(a,b)+N(a,b)],$$
where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Theorem 3.8

Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic nonconvex functions. If $fg\in L[a,b]$, then
$$\int_a^b\int_0^1 f\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)\mathrm{d}t\mathrm{d}x$$
$$\le\frac{1}{3}\int_a^b f(x)g(x)\mathrm{d}x+\frac{1}{12}(b-a)\left[M(a,b)+N(a,b)\right]$$
$$+\frac{(b^p-a^p)x^{p+1}}{12pa^pb^p}\left[M(a,b)+N(a,b)\right],$$
where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Proof. Let f,g be two harmonic nonconvex functions on I. Then
$$f\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)$$
$$\le(1-t)f(x)+tf\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right),$$
$$g\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)$$
$$\le(1-t)g(x)+tg\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right),$$
$$\forall x,[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\in I,t\in[0,1].$$
Thus
$$f\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)$$
$$\le\left[(1-t)f(x)+tf\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\right]\left[(1-t)g(x)+tg\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\right]$$
$$=(1-t)^2f(x)g(x)+t(1-t)\left[\begin{array}{l}f(x)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\\+g(x)f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\end{array}\right]$$
$$+t^2f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)$$
Integrating over [0, 1], we have

$$\int_0^1 f\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)\mathrm{d}t$$

$$\leq f(x)g(x)\int_0^1(1-t)^2\mathrm{d}t + f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\int_0^1 t^2\mathrm{d}t + \left[\begin{array}{l} f(x)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right) \\ +g(x)f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\end{array}\right]\int_0^1 t(1-t)\mathrm{d}t$$

$$=\frac{1}{3}\left[\begin{array}{l} f(x)g(x) \\ +f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\end{array}\right] + \frac{1}{6}\left[\begin{array}{l} f(x)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right) \\ +g(x)f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\end{array}\right]$$

Now, integrating over $[a,\ b]$, we have

$$\int_a^b\int_0^1 f\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)\mathrm{d}t\mathrm{d}x$$

$$\leq \frac{1}{3}\int_a^b f(x)g(x)\mathrm{d}x + \frac{1}{3}\int_a^b f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\mathrm{d}x + \frac{1}{6}\left[\begin{array}{l} g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\int_a^b f(x)\mathrm{d}x \\ +f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\int_a^b g(x)\mathrm{d}x\end{array}\right]$$

$$\leq \frac{1}{3}\int_a^b f(x)g(x)\mathrm{d}x + \frac{1}{12}(b-a)\left(f(a)+f(b)\right)\left(g(a)+g(b)\right) + \frac{2(b^p-a^p)x^{p+1}}{6pa^pb^p}\left[\begin{array}{l}\left(\frac{g(a)+g(b)}{2}\right) \\ \left(\frac{f(a)+f(b)}{2}\right)\end{array}\right]$$

$$=\frac{1}{3}\int_a^b f(x)g(x)\mathrm{d}x + \frac{1}{12}(b-a)\left[M(a,b)+N(a,b)\right] + \frac{(b^p-a^p)x^{p+1}}{12pa^pb^p}\left[M(a,b)+N(a,b)\right],$$

which is the required result.

If $p=1$, then Theorem 3.8 reduces to the following result which appears to be new one.

Corollary 3.12. Let $f,g: I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic convex functions. If $fg \in L[a,b]$, then

$$\int_a^b\int_0^1 f\left(\frac{x[\frac{2ab}{a+b}]}{tx+(1-t)[\frac{2ab}{a+b}]}\right) g\left(\frac{x[\frac{2ab}{a+b}]}{tx+(1-t)[\frac{2ab}{a+b}]}\right)\mathrm{d}t\mathrm{d}x$$
$$\leq \frac{1}{3}\int_a^b f(x)g(x)\mathrm{d}x + \frac{1}{12}(b-a)\left[M(a,b)+N(a,b)\right]$$
$$+\frac{(b-a)x^2}{12ab}\left[M(a,b)+N(a,b)\right],$$

where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

If $p=-1$, then Theorem 3.8 reduces to the following result.

Corollary 3.13. [3]. Let $f,g: I \subset \mathbb{R} \to \mathbb{R}$ be convex functions. If $fg \in L[a,b]$, then

$$\int_a^b\int_0^1 f\left((1-t)x+t[\frac{a+b}{2}]\right) g\left((1-t)x+t[\frac{a+b}{2}]\right)\mathrm{d}t\mathrm{d}x$$
$$\leq \frac{1}{3}\int_a^b f(x)g(x)\mathrm{d}x + \frac{b-a}{6}\left[M(a,b)+N(a,b)\right],$$

where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Theorem 3.9. Let $f,g: I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be similarly ordered harmonic nonconvex functions. If $fg \in L[a,b]$, then

$$\int_a^b\int_0^1 f\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)$$
$$g\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)\mathrm{d}t\mathrm{d}x$$
$$\leq \frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x + \frac{1}{8}(b-a)\left[M(a,b)+N(a,b)\right],$$

where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Proof. Let f, g be two similarly ordered harmonic nonconvex functions on I. Thus, from (3.9)

$$f\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)$$
$$\times g\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)$$
$$\leq (1-t)f(x)g(x)$$
$$+tf\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)$$

Now, integrating over [a, b], we have

$$\int_a^b\int_0^1 f\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)$$
$$\times g\left(\left[\frac{x^p[\frac{2a^pb^p}{a^p+b^p}]}{tx^p+(1-t)[\frac{2a^pb^p}{a^p+b^p}]}\right]^{\frac{1}{p}}\right)\mathrm{d}t\mathrm{d}x$$
$$\leq \frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x$$
$$+\frac{1}{2}\int_a^b f\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)g\left([\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}\right)\mathrm{d}x$$
$$\leq \frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x$$
$$+\frac{1}{8}(b-a)\big(f(a)+f(b)\big)\big(g(a)+g(b)\big)$$

$$=\frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x$$
$$+\frac{1}{8}(b-a)\left[M(a,b)+N(a,b)\right],$$
which is the required result.

Theorem 3.10. Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic nonconvex functions. If $fg\in L[a,b]$, then

$$\frac{pa^{2p}b^p}{(b^p-a^p)^2}\int_a^b\frac{b^p-x^p}{x^{1+2p}}\left[f(a)g(x)+g(a)f(x)\right]\mathrm{d}x$$
$$+\frac{pa^pb^{2p}}{(b^p-a^p)^2}$$
$$\times\int_a^b\frac{x^p-a^p}{x^{1+2p}}\left[f(b)g(x)+g(b)f(x)\right]\mathrm{d}x$$
$$\le\frac{M(a,b)}{3}+\frac{N(a,b)}{6}$$
$$+\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x,$$
where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Proof. Let f, g be two harmonic nonconvex functions. Then

$$f\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)\le(1-t)f(a)+tf(b),$$
$$g\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)\le(1-t)g(a)+tg(b),$$
$\forall a,b\in I, t\in[0,1]$.

Now, using $\langle x_1-x_2,x_3-x_4\rangle\ge 0,(x_1,x_2,x_3,x_4\in\mathbb{R})$ and $x_1<x_2,\ x_3<x_4,$ we have

$$f\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)[(1-t)g(a)+tg(b)]$$
$$+g\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)[(1-t)f(a)+tf(b)]$$
$$\le[(1-t)f(a)+tf(b)][(1-t)g(a)+tg(b)]$$
$$+f\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)g\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)$$
and we obtain
$$g(a)(1-t)f\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)$$
$$+g(b)tf\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)$$
$$+f(a)(1-t)g\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)$$
$$+f(b)tg\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)$$
$$\le(1-t)^2f(a)g(a)+t^2f(b)g(b)$$
$$+t(1-t)\left[f(a)g(b)+f(b)g(a)\right]$$
$$+f\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)g\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)$$
Integrating over [0, 1], we have
$$g(a)\int_0^1(1-t)f\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)\mathrm{d}t$$
$$+g(b)\int_0^1 tf\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)\mathrm{d}t$$
$$+f(a)\int_0^1(1-t)g\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)\mathrm{d}t$$
$$+f(b)\int_0^1 tg\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)\mathrm{d}t$$
$$\le f(a)g(a)\int_0^1(1-t)^2\mathrm{d}t+f(b)g(b)\int_0^1t^2\mathrm{d}t$$
$$+\left[f(a)g(b)+f(b)g(a)\right]\int_0^1t(1-t)\mathrm{d}t$$
$$+\int_0^1f\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)g\left([\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}\right)\mathrm{d}t$$
This implies

$$\frac{pa^{2p}b^{p}}{(b^{p}-a^{p})^{2}}\int_{a}^{b}\frac{b^{p}-x^{p}}{x^{1+2p}}\left[f(a)g(x)+g(a)f(x)\right]\mathrm{d}x$$
$$+\frac{pa^{p}b^{2p}}{(b^{p}-a^{p})^{2}}\int_{a}^{b}\frac{x^{p}-a^{p}}{x^{1+2p}}\left[f(b)g(x)+g(b)f(x)\right]\mathrm{d}x$$

$$\leq\frac{M(a,b)}{3}+\frac{N(a,b)}{6}+\frac{pa^{p}b^{p}}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x,$$
which is the required result.

If $p=1$, then Theorem 3.10 reduces to the following result which appears to be new one.

Corollary 3.14. Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic convex functions. If $fg\in L[a,b]$, then

$$\frac{a^{2}b}{(b-a)^{2}}\int_{a}^{b}\frac{b-x}{x^{3}}\left[f(a)g(x)+g(a)f(x)\right]\mathrm{d}x$$
$$+\frac{ab^{2}}{(b-a)^{2}}\int_{a}^{b}\frac{x-a}{x^{3}}\left[f(b)g(x)+g(b)f(x)\right]\mathrm{d}x$$
$$\leq\frac{M(a,b)}{3}+\frac{N(a,b)}{6}+\frac{ab}{b-a}\int_{a}^{b}\frac{f(x)g(x)}{x^{2}}\mathrm{d}x,$$
where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

If $p=-1$, then Theorem 3.10 reduces to the following result.

Corollary 3.15. [19]. Let $f,g:I\subset\mathbb{R}\to\mathbb{R}$ be convex functions. If $fg\in L[a,b]$, then

$$\frac{1}{(b-a)^{2}}\int_{a}^{b}(b-x)\left[f(a)g(x)+g(a)f(x)\right]\mathrm{d}x$$
$$+\frac{1}{(b-a)^{2}}\int_{a}^{b}(x-a)\left[f(b)g(x)+g(b)f(x)\right]\mathrm{d}x$$
$$\leq\frac{M(a,b)}{3}+\frac{N(a,b)}{6}+\frac{1}{b-a}\int_{a}^{b}f(x)g(x)\mathrm{d}x,$$
where $M(a,b)$ and $N(a,b)$ are given by (3.12) and (3.13).

Theorem 3.11 Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic nonconvex functions. If $fg\in L[a,b]$, then

$$\frac{pa^{p}b^{p}}{b^{p}-a^{p}}\int_{a}^{b}\begin{bmatrix}f\left(\left[\frac{2a^{p}b^{p}}{a^{p}+b^{p}}\right]^{\frac{1}{p}}\right)\frac{g(x)}{x^{p+1}}\\+g\left(\left[\frac{2a^{p}b^{p}}{a^{p}+b^{p}}\right]^{\frac{1}{p}}\right)\frac{f(x)}{x^{p+1}}\end{bmatrix}\mathrm{d}x$$
$$\leq\frac{pa^{p}b^{p}}{2(b^{p}-a^{p})}\int_{a}^{b}\frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x+\frac{1}{12}M(a,b)$$
$$+\frac{1}{6}N(a,b)+f\left(\left[\frac{2a^{p}b^{p}}{a^{p}+b^{p}}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{2a^{p}b^{p}}{a^{p}+b^{p}}\right]^{\frac{1}{p}}\right),$$
where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

Proof. Let f,g be two harmonic nonconvex functions with $t=\frac{1}{2}$. Then

$$f\left(\left[\frac{2x^{p}y^{p}}{x^{p}+y^{p}}\right]^{\frac{1}{p}}\right)\leq\frac{f(x)+f(y)}{2},\qquad\forall x,y\in I,$$
$$g\left(\left[\frac{2x^{p}y^{p}}{x^{p}+y^{p}}\right]^{\frac{1}{p}}\right)\leq\frac{g(x)+g(y)}{2},\qquad\forall x,y\in I.$$
Let

$$x=\left[\frac{a^{p}b^{p}}{ta^{p}+(1-t)b^{p}}\right]^{\frac{1}{p}},\text{ and }y=\left[\frac{a^{p}b^{p}}{(1-t)a^{p}+tb^{p}}\right]^{\frac{1}{p}}.$$
Thus

$$f\left(\left[\frac{2a^{p}b^{p}}{a^{p}+b^{p}}\right]^{\frac{1}{p}}\right)\leq\frac{1}{2}\begin{bmatrix}f\left(\left[\frac{a^{p}b^{p}}{ta^{p}+(1-t)b^{p}}\right]^{\frac{1}{p}}\right)\\+f\left(\left[\frac{a^{p}b^{p}}{(1-t)a^{p}+tb^{p}}\right]^{\frac{1}{p}}\right)\end{bmatrix},$$
$$g\left(\left[\frac{2a^{p}b^{p}}{a^{p}+b^{p}}\right]^{\frac{1}{p}}\right)\leq\frac{1}{2}\begin{bmatrix}g\left(\left[\frac{a^{p}b^{p}}{ta^{p}+(1-t)b^{p}}\right]^{\frac{1}{p}}\right)\\+g\left(\left[\frac{a^{p}b^{p}}{(1-t)a^{p}+tb^{p}}\right]^{\frac{1}{p}}\right)\end{bmatrix}.$$
Now, using $\langle x_1-x_2,x_3-x_4\rangle\geq0,(x_1,x_2,x_3,x_4\in\mathbb{R})$ and $x_1<x_2,\ x_3<x_4,$ we have

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$$\frac{1}{2}f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\left[\begin{array}{l} g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right) \\ +g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\end{array}\right]$$

$$+\frac{1}{2}g\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\left[\begin{array}{l} f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right) \\ +f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\end{array}\right]$$

$$\leq\frac{1}{4}\left\{\begin{array}{l}\left[f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\right] \\ \left[g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)+g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\right]\end{array}\right\}$$

$$+f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)$$

$$=\frac{1}{4}\left[\begin{array}{l} f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right) \\ +f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right) \\ +f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right) \\ +f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\end{array}\right]$$

$$+f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)$$

$$\leq\frac{1}{4}\left[\begin{array}{l} f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right) \\ +f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right) \\ +\left[(1-t)f(a)+tf(b)\right]\left[tg(a)+(1-t)g(b)\right] \\ +\left[tf(a)+(1-t)f(b)\right]\left[(1-t)g(a)+tg(b)\right]\end{array}\right]$$

$$+f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)$$

$$\frac{1}{4}\left[\begin{array}{l} f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right) \\ +f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right) \\ +2t(1-t)\left[f(a)g(a)+f(b)g(b)\right] \\ +\left[t^2+(1-t)^2\right]\left[f(a)g(b)+f(b)g(a)\right]\end{array}\right]$$

$$+f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)$$

Integrating over [0, 1], we have

$$\frac{1}{2}f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\int_0^1\left[\begin{array}{l} g\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right) \\ +g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\end{array}\right]\mathrm{d}t$$

$$\frac{1}{2}g\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\int_0^1\left[\begin{array}{l} f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right) \\ +f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\end{array}\right]\mathrm{d}t$$

$$\leq \frac{1}{4}\left[\begin{array}{l} \int_0^1 f\left(\left[\frac{a^p b^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{a^p b^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ +\int_0^1 f\left(\left[\frac{a^p b^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{a^p b^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ +\left[f(a)g(a)+f(b)g(b)\right]\int_0^1 2t(1-t)\mathrm{d}t \\ +\left[f(a)g(b)+f(b)g(a)\right]\int_0^1\left[t^2+(1-t)^2\right]\mathrm{d}t \end{array}\right]$$

$$+f\left(\left[\frac{2a^p b^p}{a^p+b^p}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{2a^p b^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)$$

$$=\frac{1}{4}\left[\begin{array}{l} \int_0^1 f\left(\left[\frac{a^p b^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{a^p b^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ +\int_0^1 f\left(\left[\frac{a^p b^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{a^p b^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ +\frac{1}{3}\left[f(a)g(a)+f(b)g(b)\right] \\ +\frac{2}{3}\left[f(a)g(b)+f(b)g(a)\right] \end{array}\right]$$

$$+f\left(\left[\frac{2a^p b^p}{a^p+b^p}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{2a^p b^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)$$

From the above inequality, it follows that

$$\frac{pa^p b^p}{b^p-a^p}\int_a^b\left[\begin{array}{l} f\left(\left[\frac{2a^p b^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\frac{g(x)}{x^{p+1}} \\ +g\left(\left[\frac{2a^p b^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\frac{f(x)}{x^{p+1}} \end{array}\right]\mathrm{d}x$$

$$\leq \frac{pa^p b^p}{2(b^p-a^p)}\int_a^b \frac{f(x)g(x)}{x^{p+1}}\mathrm{d}x+\frac{1}{12}M(a,b)$$

$$+\frac{1}{6}N(a,b)+f\left(\left[\frac{2a^p b^p}{a^p+b^p}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{2a^p b^p}{a^p+b^p}\right]^{\frac{1}{p}}\right),$$

which is the required result..

If *p=1*, then Theorem 3.11 reduces to the following result which appears to be new one.

Corollary 3.16. Let $f,g: I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic convex functions. If $fg\in L[a,b]$, then

$$\frac{ab}{b-a}\int_a^b\left[f\left(\frac{2ab}{a+b}\right)\frac{g(x)}{x^2}+g\left(\frac{2ab}{a+b}\right)\frac{f(x)}{x^2}\right]\mathrm{d}x$$

$$\leq \frac{ab}{2(b-a)}\int_a^b\frac{f(x)g(x)}{x^2}\mathrm{d}x+\frac{1}{12}M(a,b)$$

$$+\frac{1}{6}N(a,b)+f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right),$$

where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

If *p=-1*, then Theorem 3.11 reduces to the following result.

Corollary 3.17. [19]. Let $f,g: I\subset\mathbb{R}\to\mathbb{R}$ be convex functions. If $fg\in L[a,b]$, then

$$\frac{1}{b-a}\int_a^b\left[f\left(\frac{a+b}{2}\right)g(x)+g\left(\frac{a+b}{2}\right)f(x)\right]\mathrm{d}x$$

$$\leq \frac{1}{2(b-a)}\int_a^b f(x)g(x)\mathrm{d}x+\frac{1}{12}M(a,b)$$

$$+\frac{1}{6}N(a,b)+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right),$$

where $M(a,b)$ and $N(a,b)$ are given by (3.11) and (3.12).

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