MAGNT Research Report (ISSN. 1444-8939) Vol.4 (1). PP. 41-52

Some Integral Transforms for Certain Starlike Functions

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Abstract: In this paper, we define and study a new subclass of starlike functions by using fractional derivative operator. Some basic properties of this class are investigated. This behavior of the functions in this class is examined under generalized type of integral transforms. Several known or new special cases of our results are also pointed out.

Keywords: Fractional derivative, starlike convex, univalent, linear integral transform**.**

1. Introduction

Let A be the class of all functions of the for

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.10$$

which are analytic in the open unit disc $E = \{z : z \in A : |z| < 1.\}$ Also let $S,\ S^*(r)\ and\ \ C(r)$ denote the subclasses of A consisting of functions which are, respectively, univalent, starlike and convex of order $r,\ \ 0 \le r < 1,\ in\ \ E.$

Let $f, g \in A,\ \ f(z)$ given by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n ,$$

then Hadamard product (or convolution) of f and g is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} b_n a_n z^n = (g * f)(z).$$

If f and g are analytic in E, then we say that the function $f(z)$ is subordinate to $g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)),\ \ z \in E.$ We denote this subordination by $f \prec g,$ or $f(z) \prec g(z).$

In particular, when $g(z)$ is univalent, then this subordination is equivalent to $f(0) = g(0)$ and $f(E) \prec g(E).$

Let $\phi(a, c : z)$ be the incomplete beta function defined as follow:

$$\phi(a, c : z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad z \in E,$$

$$c \neq 0, -1, -2, -3, ...,$$

where $(\rho)_n$ Pochhamer symbol defined in terms of Gamma function Γ by

$$(\rho)_n = \frac{\Gamma(n+\rho)}{\Gamma(\rho)}.$$

In [2], an operator $\Im : A \to A$ is defined as

$$\Im(a, c) = \phi(a, c : z) * f(z),\ \ z \in E.$$

The fractional derivative of order β is defined for a function $f(z)$ by

$$D_z^{\beta} f(z) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\beta}} d\xi,\ \ 0 \le \beta < 1,$$

where $f(z)$ is an analytic function in a simply connected domain of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\beta}$ is removed by requiring $\log(z-\xi)$ to be real, when $(z-\xi) > 0$, see [16].

Using $D_z^{\beta} f(z)$, Owa and Srivastava [9] introduced the operator $\Im_{\beta} : A \to A,$ known as the extension of fractional derivative and fractional integral as follows:

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)

$$\Im_\beta f(z) = \Gamma(2-\beta) z^\beta D_z^\beta f(z)$$
$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\beta)}{\Gamma(k+1-\beta)} a_k z^k$$
$$= \phi(2, 2-\beta); z) * f(z), \ \beta \neq 2, 3, \ldots,$$

where $\Im_0 f(z) = f(z)$.

Let a linear multiplier fractional operator $\Im_\lambda^\beta : A \to A$ be defined as follows

$$\Im_\lambda^\beta f(z) = (1-\lambda)\Im_\beta f(z) + \lambda z(\Im_\beta f(z))', \ \lambda \geq 0$$
$$= \phi_\lambda(z) * \Im_\beta f(z), \qquad (1.2)$$

where $\phi_\lambda(z) = \dfrac{z-(1-\kappa)z^2}{(1-z)^2}$.

Our work is motivated and related to a one-parameter family of functions defined and discussed in [15] by Sokol. We give some of the properties of these functions, which we shall need in our work.

Let, for $\alpha \in (-3, 1)$, $z \in E$,

$$\tilde{p}_\alpha(z) = \frac{1}{3+(\alpha-3)z+\alpha z^2}$$
$$= \frac{3}{3+\alpha}\left[\frac{1}{1-z} + \frac{\alpha}{\alpha z +}\right]. \qquad (1.3)$$

Then

(i) $\tilde{p}_\alpha(z)$ is univalent in E.

(ii) $\tilde{p}_\alpha(z) = 1 + \dfrac{(3-\alpha)^2}{3(3+\alpha)} + \ldots$

$$\frac{9(1+\alpha)}{2(3+\alpha)^2} \leq \Re \tilde{p}_\alpha(z)$$

(iii) $$= \Re\left[\frac{-3}{(z-1)(\alpha z+3)}\right]$$
$$\leq \frac{3}{2(3-\alpha)}.$$

(iv) For $-3 < \alpha_1 < \alpha_2 < 1$, $\tilde{p}_{\alpha_2}(z) < \tilde{p}_{\alpha_1}(z)$.

(v) For $\alpha \in [-1, 1]$, $\tilde{p}_\alpha(z)$ is convex univalent in E.

Definition 1.1. Let $p(z)$ be analytic in E. with $p(0) = 1$.

Then $p \in P(\tilde{p}_\alpha)$, *if* $p(z) < \tilde{p}_\alpha(z)$, where $\tilde{p}_\alpha(z)$ is given by (1.3) and $z \in E$.

We note that, for $\alpha \in [-1, 1]$,

$$P(\tilde{p}_\alpha) \subset P(\gamma) \subset P(0) = P,$$

where $P(\gamma)$ consists of analytic functions with positive real part greater than γ, and

$$\gamma = \frac{9(1+\alpha)}{2(3+\alpha)^2} \in [0, \frac{9}{16}]. \qquad (1.4)$$

Definition1.2. Let $f \in A$. Then $f \in \tilde{S}_\lambda(\alpha, \beta)$ if and only if

$$\frac{z(\Im_\lambda^\beta f(z))'}{\Im_\lambda^\beta f(z)} \in P(\tilde{p}_\alpha), \ \ z \in E, \ \ \alpha \in (-3, 1)$$

and the operator $\Im_\lambda^\beta$ is defined by (1.2)

The corresponding class $\tilde{C}_\kappa(\alpha, \beta)$ is defined by the relation

$$f \in \tilde{C}_\lambda(\alpha, \beta) \Leftrightarrow zf' \in \tilde{S}_\lambda(\alpha, \beta), \ \ z \in E. \qquad (1.5)$$

For $\alpha \in [-1, 1]$, *and* $\beta = \lambda = 0$,

$$\tilde{S}_0(\alpha, 0) = \tilde{S}(\alpha) \subset S^*(\gamma)$$
$$\tilde{C}_\alpha(\alpha, 0) = \tilde{C}(\alpha) \subset C(\gamma),$$

where γ is given by (1.4).

2. Preliminary Results

We shall need the following Lemmas in our investigation.

Lemma 2.1.{3]. Let η_1 *and* μ_1 be complex constants and $h(z)$ be a complex univalent functions in E with

$$h(0) = 1, \ \Re\left\{p(z) + \frac{zp'(z)}{\eta_1 p(z) + \mu_1}\right\} > 0.$$

Suppose $p \in P$ satisfies the differential equation

$$\left\{p(z) + \frac{zp'(z)}{\eta_1 p(z) + \mu_1}\right\} \prec h(z), \ z \in E. \qquad (2.1).$$

If the differential equation

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)

$$q(z)+\frac{zq'(z)}{\eta_1 q(z)+\mu_1}\prec h(z),\quad q(0)=1 \qquad (2.2)$$

has univalent solution $q(z)$ in E, then

$$p(z)\prec q(z)\prec h(z)$$

and $q(z)$ is the best dominant in (2.2).

Lemma 2.2.[11]. Let $f(z)$ *and* $g(z)$ be in class $C(0)=C$ and $S^*(0)=S^*$ respectively. Then , for every function $F(z)$ analytic in E, with $F(0)=1$, we have

$$\frac{F(z)*g(z)F(z)}{f(z)*g(z)}\in \bar{C}_o(F(E)),\; z\in E,$$

where $\bar{C}_o$ denotes the closed convex hull.

The same result holds if both $f(z)$ and $g(z)$ are starlike of order $\frac{1}{2}$.

Lemma 2.3 [6]. Let $u=u_1+iu_2$ and $v=v_1+iv_2$ and $\psi(u,v)$ be a complex-valued function satisfying the conditions:

i. $\psi(u,v)$ is continuous in a domain $D\subset\mathbb{C}^2$.

ii. $(0,1)\in D$ and $\Re\psi(1,0)>0$.

iii. $\Re\psi(u,v)\le 0$, whenever $(iu_2,v_1)\in D$ and $v_1\le\frac{-1}{2}(1+u_2^2)$.

If $h(z)=1+\sum_{n+1}^{\alpha}c_n z^n$ is analytic in E such that $(h(z),zh'(z))\in D$ and

$$\Re\{\psi(h(z),zh'(z))\}>0,\; z\in E,$$

then $\Re\{h(z)\}>0$ in E.

Lemma 2.4 [5]. Let $0\le\beta_1\le\beta_2\le 1$. Then $\phi(2-\beta,2-\beta_1:z)\in S^*(\frac{1}{2})$.

Lemma 2.5 [13]. Let $p(z)$ be an analytic function in E with $p(0)=1$ and $\Re\{p(z)\}>0,\; z\in E$. Then, for $s>0$ and

$$\xi\neq -1(complex),\; \Re\left\{p(z)+\frac{szp'(z)}{p(z)+\xi}\right\}>0$$

for $|z|<r_0$, where r_0 is given by

$$r_0=\frac{|\xi+1|}{\sqrt{A+(A^2-|\xi^2-1|^2)^{\frac{1}{2}}}}, \qquad (2.3)$$

$$A=2(s+1)^2+|\varsigma|^2-1,$$

and this radius Is best possible.

3. Basic Results

In this section, we shall assume

$$\lambda>0,\; \alpha\in[-1,1],\; \gamma=\frac{9(1+\alpha)}{2(3+\alpha)^2},$$

unless otherwise stated.

Theorem3.1 . Let $f\in\tilde{S}_\lambda(\alpha,\beta)$, for $\beta\in[0,1]$. Then $f\in\tilde{S}(\alpha)$ for $z\in E$.

Proof. Let $\psi_\lambda(z)=[\phi_\lambda(z)]^{-1}$, where

$$\phi_\lambda(z)=\frac{z[1-(1-\lambda)z]}{(1-z)^2}.$$

Then

$$\frac{\psi_\lambda(z)}{z}=1+\sum_{n=1}^{\infty}\frac{z^n}{n\lambda+1}=\int_0^1\frac{dt}{1-zt^\lambda},$$

where ψ_λ is convex and

$$\Re\frac{\psi_\lambda(z)}{z}\ge\int_0^1\frac{dt}{1+t^\lambda}=\delta,\quad \frac{1+\lambda}{2+\lambda}\le\delta<1.$$

see[10].

Now

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)

$$\frac{zf'(z)}{z}$$
$$=\frac{z[\phi(2-\beta,2;z)*\phi(2,2-\beta;z)]*\phi_\lambda(z)*\psi_\lambda(z)*f'(z)}{\phi(2-\beta,2;z)*\phi(2,2-\beta;z)*\phi_\lambda(z)*\psi_\lambda(z)*f(z)}$$
$$=\frac{\{\phi(2-\beta,2;z)*\psi_\lambda(z)\}*[z(\phi(2,2-\beta))*\phi_\lambda(z)*f(z)]'}{\{\phi(2-\beta,2;z)*\psi_\lambda(z)\}*[\phi(2,2-\beta)*\phi_\lambda(z)*f(z)]}$$
$$=\frac{\phi_*(z)*\dfrac{z(\Im_\lambda^\beta f(z))'}{\Im_\lambda^\beta f(z)}\Im_\lambda^\beta f(z)}{\phi_*(z)*\Im_\lambda^\beta f(z)},\ \phi_*=\phi(2-\beta,2;z)*\psi_\lambda$$
$$=\frac{\phi_*(z)*(f(z)\Im_\lambda^\beta f(z))}{\phi_*(z)*\Im_\lambda^\beta f(z)}. \tag{3.1}$$

It can easily be seen that $\phi(2-\beta,2;z)$ is a convex function, since

$$z\phi'(2-\beta,2;z)=\phi(2-\beta,2;z)$$
$$=\frac{z}{(1-z)^{2-\beta}}\in S^*(\frac{\beta}{2})\subset S^*.$$

Also $\psi'(z)$ is convex and therefore $\phi_*(z)$ is also convex in E, see[12]. Since $f\in\tilde{S}_\lambda(\alpha,\beta)$, $\Im_\lambda^\beta f\in\tilde{S}(\alpha)\subset S^*$, for $\alpha\in[-1,1]$, see[15] and $F\in P(\tilde{p}_\alpha)$, we apply Lemma 2.2 to obtain the required result from(1.3) that $f\in\tilde{S}(\alpha)$ in E. This complete the proof. □

We note that, for $\lambda\geq 0$,
$\tilde{S}(\alpha,\beta)\subset\tilde{S}(\alpha)\subset S^*(\gamma),\quad \beta\in[0,1].$
Using Theorem 3.1 and the relation (1.5), we can easily prove the following:
$\tilde{C}(\alpha,\beta)\subset\tilde{C}(\alpha)\subset C(\gamma),\quad \beta\in[0,1].$

Theorem 3.2. Let $0\leq\beta_1\leq\beta<1$.Then, for $z\in E$,

(i). $\tilde{S}_\lambda(\alpha,\beta)\subset\tilde{S}_\lambda(\alpha,\beta_1)$.

(ii) $\tilde{C}_\lambda(\alpha,\beta)\subset\tilde{C}_\lambda(\alpha,\beta_1)$.

Proof. (i). Let $f\in\tilde{S}_\lambda(\alpha,\beta)$. Then

$$\Im_\lambda^{\beta_1} f(z)$$
$$=[\phi(2,2-\beta_1;z)*\phi_\lambda(z)]*f(z)$$
$$=[\phi(2-\beta,2-\beta_1;z)]*[\phi_\lambda(z)*\phi(2,2-\beta;z)*f(z)]$$
$$=\phi(2-\beta,2-\beta_1;z)]*\Im_\lambda^\beta f(z),$$

and

$$z(\Im_\lambda^{\beta_1} f(z))'=\phi(2-\beta,2-\beta_1;z)*z(\Im_\lambda^\beta f(z))',$$

where

$\phi(2-\beta,2-\beta_1;z)\in C$, see[5].

Now

$$\frac{z(\Im_\lambda^{\beta_1} f(z))'}{\Im_\lambda^{\beta_1} f(z)}$$
$$=\frac{\phi(2-\beta,2-\beta_1;z)*\left(\dfrac{z(\Im_\lambda^{\beta_1} f(z))'}{\Im_\lambda^{\beta_1} f(z)}\Im_\lambda^\beta f(z)\right)}{\phi(2-\beta,2-\beta_1;z)*\Im_\lambda^\beta f(z)}$$
$$=\frac{\phi(2-\beta,2-\beta_1;z)*(F\Im_\lambda^\beta f(z))}{\phi(2-\beta,2-\beta_1;z)*\Im_\lambda^\beta f(z)}.$$

Since $f\in\tilde{S}_\lambda(\alpha,\beta)$, it follows that $F\in P(\tilde{p}_\alpha)$ and

$$\Im_\lambda^\beta f\in\tilde{S}(\alpha)\subset S^*(\gamma)\subset S^*.$$

Thus, using Lemma 2.2, $\Im_\lambda^{\beta_1} f\in\tilde{S}(\alpha\ )$

This completes the proof that $f\in\tilde{S}_\lambda(\alpha,\beta)$ in E. The proof of (ii) follows easily from (i) and relation (1.5). □

Theorem3.3. $\tilde{S}_\lambda(\alpha,\beta)\subset\tilde{S}_0(\alpha,\beta)$, $\lambda>0$.

Proof. Let $f\in\tilde{S}_\lambda(\alpha,\beta)$, Set

$$\frac{z(\phi(2,2-\beta;z)*f(z))'}{\phi(2,2-\beta;z)*f(z)}=p(z),$$

where $p(z)$ is analytic in E with $p(0)=1$. The some simple computations give us

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)

$$(1-\lambda)\frac{z(\phi(2,2-\beta;z)*f(z))'}{\phi(2,2-\beta;z)*f(z)} + \lambda\frac{]z(\phi(2,2-\beta;z)*f(z))']'}{[\phi(2,2-\beta;z)*f(z)]'} = p(z)+\lambda\frac{zp'(z)}{p(z)}$$

and so

$$p(z)+\frac{zp'(z)}{\frac{1}{\lambda}p(z)} \prec \tilde{p}_\alpha(z) \quad \text{in} \quad E.$$

Since $\Re\tilde{p}_\alpha(z)<r$, $r\in[0,1]$, we use lemma 2.1 to have $p(z)\prec\tilde{p}_\alpha(z)$ in E. This proves that $f\in\tilde{S}_\lambda(\alpha,\beta)$ in E. □

Remark 3.1. From the properties of the class $P(\tilde{p}_\alpha)$, we can easily prove the following inclusions results.

(i). For $-1\le\alpha_1<\alpha_2<1$,

$$\tilde{S}_\lambda(\alpha_2,\beta)\subset\tilde{S}_\lambda(\alpha_1,\beta)$$

(ii). For $-1\le\lambda_1<\lambda_2<1$,

$$\tilde{S}_{\lambda_2}(\alpha_2,\beta)\subset\tilde{S}_{\lambda_1}(\alpha_1,\beta).$$

Remark 3.2. It is shown in [15] that

(i). $C(\gamma_1)\subset\tilde{S}(\alpha)$, $\alpha\in[-3,1)$,

and $\gamma_1=\frac{3\alpha}{2(3-\alpha).}$.

(ii). $\tilde{S}(\alpha)\subset S^*(\gamma)$, $\alpha\in[-1,1]$.

Using Remark 3.2, we can easily prove the following.

Theorem 3.4. Let, for $-3\le\alpha<1$, $\Re\frac{z(\Im_1^\beta f(z))'}{\Im_1^\beta f(z)}>\frac{3\alpha}{2(3-\alpha)}$.

Then

$$\Re\frac{z(\Im_0^\beta f(z))'}{\Im_0^\beta f(z)}>\gamma,\quad z\in E.$$

Proof. The proof is immediately, when we note that

$$\Im_1^\beta f(z)=\Im_\beta f(z)*\frac{z}{(1-z)^2}=z(\Im_\beta f(z))'$$

and

$$\Im_0^\beta f(z)=\Im_\beta f(z)*\frac{z}{1-z}=\Im_\beta f(z).$$

Theorem 3.5. Let $f\in\tilde{S}_\lambda(\alpha,\beta)$ and be given by (1.1). Then

$$|a_2|\le\frac{(1-\gamma)\Gamma(3-\beta)}{(1+\lambda)\Gamma(2-\beta)}.$$

Proof. Set

$$.\ \frac{z(\Im_\lambda^\beta f(z))'}{\Im_\lambda^\beta f(z)}=p(z)=1+\sum_{n=1}^{\infty}c_nz^n, \qquad (3.2)$$

where $p\in P(\tilde{p}_\alpha)\subset P(\gamma)$.

From the definition of $\Im_\lambda^\beta f(z)$, we can write

$$\Im_\lambda^\beta f(z)=z+\sum_{n=2}^{\infty}\Psi_n(\beta,\lambda)a_nz^n, \qquad (3.3)$$

where

$$\Psi_n(\beta,\lambda)=\frac{\Gamma(n+1)\Gamma(2-\beta)}{\Gamma(n+1-\beta)}(1+\lambda(n-1)). \qquad (3.4)$$

Using (3.3), (3.4) in (3.2) and computing coefficients of z^n on both sides, we get

$$(n-1)a_n\Psi_n(\beta,\lambda)=\sum_{j=1}^{\infty}c_{n-j}a_j\Psi_j(\beta,\lambda),\ a_1=1. \qquad (3.5)$$

From (3.5), with $n=2$, we have

$$|a_n|=\frac{1}{\Psi_2(\beta,\lambda)}|c|\le\frac{2(1-\gamma)}{\Psi_2(\beta,\lambda)},$$

where we have used the well known coefficient result $|c_n|\le2(1-\gamma)$ for functions with positive real part greater than γ.

Since

$$\Psi_2(\beta,\lambda)=\frac{\Gamma(3)\Gamma(2-\beta)}{\Gamma(3-\beta)}(1+\lambda) = 2(1+\lambda)\frac{\Gamma(2-\beta)}{\Gamma(3-\chi)},$$

we obtain the required result. □

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)

The following covering theorem is an application of Theorem 3.5.

Theorem 3.6. Let, for $\beta \in [0,1]$, $f \in \tilde{S}_\lambda(\alpha,\beta)$. Then $f(E)$ contains an open disc of radius

$$\left\{ \frac{(1+\lambda)\Gamma(2-\beta)}{2(1+\lambda)\Gamma(2-\beta)+(1-\gamma)\Gamma(3-\beta)} \right\}.$$

Proof. Since $f \in \tilde{S}_\lambda(\alpha,\beta)$ and Theorem 3.1, $\tilde{S}_\lambda(\alpha,\beta) \subset \tilde{S}(\alpha)$, where $\tilde{S}(\alpha) \subset S^*(\gamma) \subset S^*$, see [15], therefore it follows that $f(z)$ is univalent in E.

Let $f(z) \neq w_0$, $w_0 \neq 0$. Then $\frac{w_0 f(z)}{w_0 - f(z)}$ is also univalent in E.
Now

$$\frac{w_0 f(z)}{w_0 - f(z)} = z(a_2 + \frac{1}{w_0})z^2 + ...,$$

where $f(z)$ is given by (1.1) and since $|a_2 + \frac{1}{w_0}| \leq 2$, we have

$$|\frac{1}{w_0}| \leq 2 + |a_2|$$
$$\leq 2 + \frac{(1-\gamma)\Gamma(3-\beta)}{(1+\lambda)\Gamma(2-\beta)}$$
$$= \frac{2(1+\lambda)\Gamma(2-\beta)+(1-\gamma)\Gamma(3-\beta)}{(1+\lambda)\Gamma(2-\beta)}.$$

That is

$$|w_0| \geq \frac{(1+\lambda)\Gamma(2-\beta)}{2(1+\lambda)\Gamma(2-\beta)+(1-\gamma)\Gamma(3-\beta)}.$$

This completes the proof. □

We note the following special cases.
(i). $\alpha = -1 \Rightarrow \gamma = 0$. Take $\lambda = 0$, $\beta = 0$. Then $|w_0| \geq \frac{1}{4}$ and $f \in S^*$. This is a well known result for starlike functions.

(ii). Let $\alpha = 0$. Then $\gamma = \frac{1}{2}$. In this case $f \in S^*(\frac{1}{2})$ and $|w_0| \geq \frac{1}{3}$.

Theorem 3.7. The class $\tilde{S}_\lambda(\alpha,\beta)$ is closed under convex convolution.

Proof. Let $f \in \tilde{S}_\lambda(\alpha,\beta)$ and $\phi \in C$. Then

$$\frac{z(\Im_\lambda^\beta(\phi * f)(z))'}{\Im_\lambda^\beta(\phi * f)(z)}$$
$$= \frac{\phi(z-\beta,2;z) * \phi_\lambda(z) * [z(\phi * f)(z)]'}{\phi(z-\beta,2;z) * \phi_\lambda(z) * (\phi * f)(z)}$$
$$= \frac{\phi(z) * z[\phi_\lambda(z) * \phi(z-\beta,2;z) * f(z)]'}{\phi(z) * [\phi_\lambda(z) * \phi(z-\beta,2;z) * f(z)]}$$
$$= \frac{\phi(z) * \frac{z(\Im_\lambda^\beta f(z))'}{\Im_\lambda^\beta f(z)} \Im_\lambda^\beta f(z)}{\phi(z) * \Im_\lambda^\beta f(z)}$$
$$= \frac{\phi(z) * (F(z)\Im_\lambda^\beta f(z))}{\phi(z) * \Im_\lambda^\beta f(z)}.$$

Now $\phi \in C$, $\Im_\lambda^\beta f \in \tilde{S}(\alpha) \subset S^*$, we apply Lemma 2.2 and it follows that $(\phi * f) \in \tilde{S}_\lambda(\alpha,\beta)$, for $z \in E$. □

Applications of Theorem 3.7

(i). Let, for $\mu \geq 0$,

$$F_\mu(f)(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)dt, \text{ see[1]}. \quad (3.6)$$

Then, we can write (3.6) as

$$\Im_\lambda^\beta F_\mu(f) = \phi_\mu(z) * f_1(z), \quad f_1(z) = \Im_\lambda^\beta f(z),$$

where

$$\phi_\mu(z) = \sum_{n=1}^{\infty}\left(\frac{\mu+1}{\mu+n}\right)z^n,$$

and it is known that $\phi_\mu(z)$ is convex in E. Thus, from Theorem 3.7, it follows that $F_\mu \in \tilde{S}_\lambda(\alpha,\beta)$ in E.

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)

(ii). Let $F_\mu(f)$, defined by (3.6), belong to $\tilde{S}_\lambda(\alpha,\beta)$. Then $f\in\tilde{S}_\lambda(\alpha,\beta)$ for $|z|<r_\mu$, where r_μ is given by

$$r_\mu=\frac{\mu+1}{2+\sqrt{\mu^2+3}},\qquad (3.7)$$

and this radius is best possible.
In fact, from (3.6), we can write

$$f_1(z)=\frac{\mu\Im_\lambda^\beta F_\mu(z)+z(\Im_\lambda^\beta F_\mu(z))'}{\mu+1}$$
$$=(\psi_\mu*\Im_\lambda^\beta F_\mu)(z),$$

where

$$\psi_\mu(z)=\frac{1}{\mu+1}\left(\frac{\mu z}{1-z}+\frac{z}{(1-z)^2}\right)\in A.$$

It can easily be shown with some simple computations that $\psi_\mu\in C$ in $|z|<r_\mu$, and r_μ is give by (3.7). Now, appealing to Theorem 3.7, the desired result follows.

As special cases, we note that:

(i). $\mu=0\ \Rightarrow\ \Im_\lambda^\beta F_0(z)=\int_0^z\frac{f_1(t)}{t}dt$

and we have $r_0=\dfrac{1}{2+\sqrt{3}}$.

(ii). $\mu=1\ \Rightarrow\ \Im_\lambda^\beta F_1(z)=\dfrac{2}{z}\int_0^z f_1(t)dt$

with $r_1=\dfrac{1}{2}$.

Remark 3.3. In (3.6), if $f\in\tilde{S}_\lambda(\alpha,\beta)$, then we directly prove that $F_\mu(f)$ also belongs to $\tilde{S}_\lambda(\alpha,\beta)$ in E.

From (3.6), we have
$$z(\Im_\lambda^\beta F_\mu(z))'+\mu\Im_\lambda^\beta F_\mu(z)=(\mu+1)f_1(z).$$

By setting

$$h(z)=\frac{z(\Im_\lambda^\beta F_\mu(z))'}{\Im_\lambda^\beta F_\mu(z)},\quad h(0)=1,\qquad (3.8)$$

and with some computation, we obtain

$$\left[h(z)+\frac{zh'(z)}{h(z)+\mu}\right]=\frac{z(\Im_\lambda^\beta f(z))'}{\Im_\lambda^\beta f(z)}\prec\tilde{p}_\alpha(z).\qquad (3.9)$$

Now applying Lemma 2.1, we prove the result. □

This method yields the following interesting results.

(I). Let $f\in\tilde{S}_\lambda(\alpha,\beta)\subset\tilde{S}(\alpha)\subset S^*(\gamma)$.
Then $\Im_\lambda^\beta F_\mu\in S^*(\sigma)$, where

$$\sigma=\frac{2M}{N+\sqrt{N^2+M}},$$
$$M=1+2\mu\gamma,\qquad (3.10)$$
$$N=2\mu-2\gamma+1$$

Proof. In (3.8), we write
$$h(z)=(1-\sigma)h_1(z)+\sigma$$
and have from 93.9), the follwing

$$\left[(1-\sigma)h_1(z)+(\sigma-\gamma)+\frac{(1-\sigma)zh_1'(z)}{(1-\sigma)h_1'(z)+(\sigma+\mu)}\right]>0,$$

where we have used the fact that
$$P(\tilde{p}_\alpha)\subset P(\gamma).$$
We now form the funnction $\psi(u,v)$ by taking
$$u=u_1+iu_2=h_1(z),\ v=v_1+iv_2=zh_1'(z)$$
and write

$$\psi(u,v)=(1-\sigma)u+(\sigma-\gamma)+\frac{(1-\sigma)v}{(1-\sigma)u+\sigma+\mu}.$$

The first two conditions of lemma 2.3 are clearly satisfied. We verify condition (iii) as follows

$$\Re[\psi(iu_2,v_1)$$
$$=(\sigma-\gamma)+\frac{(\sigma+\mu)(1-\sigma)v_1}{(1-\sigma)^2u_2^2+(\sigma+\mu)^2}$$
$$\le(\sigma-\gamma)-\frac{1}{2}\frac{(\sigma+\mu)(1-\sigma)(1+u_2^2)}{(1-\sigma)^2u_2^2+(\sigma+\mu)^2}$$
$$=\frac{A_1+Bu_2^2}{2C},$$

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)

where

$A_1 = 2(\sigma-\gamma)(\sigma+\mu)^2 - (\sigma+\mu)(1-\sigma)$

$B = 2(\sigma-\gamma)(1-\sigma)^2 - (\sigma+\mu)(1-\sigma)$

$C = [(1-\sigma)^2 u_2^2 + (\sigma+\mu)^2] > 0.$

For $\Re\psi(iu_2, v_1) \le 0,$ both $A_1 \le 0$ and $B \le 0.$ Form this condition, we get the value of $\sigma \in (0,1)$ as given by (3.10). This proves $\Re h(z) > 0$ by applying Lemma 2.3 and therefore $\Im_\lambda^\beta F_\mu \in S^*(\sigma)$ in $E.$ □

II. Let $f \in \tilde{S}_\lambda(\alpha,\beta).$ Then $\Im_\lambda^\beta F_\mu \in S^*(\sigma_1),$ where

$$\sigma_1 = \left([\int_0^1 \frac{t^\mu}{(1+t)^2} dt]^{-1} - \mu \right).$$

Proof. $\alpha = 0 \Rightarrow \gamma = 0$ and so $\Im_\lambda^\beta f \in S^*.$

With $\frac{z(\Im_\lambda^\beta F_\mu(z))'}{\Im_\lambda^\beta F_\mu(z)} = p(z),$ we have from (3.6),

$$\left(p(z) + \frac{zp'(z)}{p(z)+\mu} \right) = \frac{z(\Im_\lambda^\beta f(z))'}{\Im_\lambda^\beta(z)} \prec \frac{1+z}{1-z}. \quad (3.11)$$

Then, proceeding as in [8], we note that the differential equation (3.11) has a univalent solution $q(z)$ with $p(z) \prec q(z) \prec \frac{1+z}{1-z}$ and $q(z)$ is the best dominant of (3.11). It is easy to verify that the solution $q(z)$ is

$$q(z) = \frac{z^{\mu+1}}{(1-z)^2 \int_0^z \frac{u^\mu}{(1-u)^2} du} - \mu$$

$$= \left[(1-z)^2 \int_0^1 \frac{t^\mu}{(1-tz)^2} du \right]^{-1} - \mu, \quad (3.12)$$

and, from (3.12), we have

$$\Re q(z) = q(-1) = \sigma_1 = \left(4\int_0^1 \frac{t^\mu}{(1+t)^2} dt \right)^{-1} - \mu.$$

It is shown in [8] that the bounds σ_1 can not be increased. This proves that $\Im_\lambda^\beta F_\mu \in S^*(\sigma_1).$

4. Integral Transformations

In this section, we show that the class $\tilde{S}_\lambda(\alpha,\beta),\ \alpha \in [-1,1]$ is persevered under some integral transforms.

Let $f \in A,\ m \in \mathbb{N} = \{1,2,3,...\},$ and let

$$F_m^1(f) = \frac{(\delta+1)^m}{z^\delta \Gamma(m)} \int_0^z \left(\log\frac{z}{t} \right)^{m-1} t^{\delta-1} f(t) dt, \quad (4.1)$$

where $\Re\delta > 0,$ and

$$F_m^2(f) = \binom{\delta+m}{\delta} \frac{m}{z^\delta} \int_0^z \left(1 - \frac{t}{z} \right)^{m-1} t^{\delta-1} f(t) dt, \quad (4.2)$$

where

$$(m-1+\Re\delta) \ge 0,\ F_0^j(f) = f(z),\ j = 1,2.$$

For δ real, $m = 1,$ both integrals given by (4.1) and (4.2) yield the integral operator (3.6). For δ real, (4.1) and (4.2) are special cases of one parameter families of linear integral transforms introduced and studied in [4].

In the following, we shall show that the class $\tilde{S}_\lambda(\alpha,\beta),\ \alpha \in [-1,1],$ is preserved under (4.1) and 94.2). This is one of main motivations of this paper.

Theorem4.1. let $f \in \tilde{S}_\lambda(\alpha,\beta),\ \alpha \in [-1,1].$ Then $F_m^j(f),$ defined by (4.1), j=1,2, also belongs to $\tilde{S}_\lambda(\alpha,\beta).$

Proof. Using beta and gamma functions and noting that

$$\binom{\eta}{\sigma} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\sigma+1)\Gamma(\sigma+1)},$$

we can write

$$F_m^1(f) = z + \sum_{k=2}^{\infty}\left(\frac{\delta+1}{\delta+k}\right)^m a_k z^k, \qquad (4.3)$$

and

$$F_m^2(f) = z + \frac{\Gamma(\delta+m+1)}{\Gamma(\delta+1)}\sum_{k=2}^{\infty}\frac{\Gamma(\delta+k)}{\Gamma(\delta+m+k)}a_k z^k. \qquad (4.4)$$

From (4.3) and (4.4), it can easily be seen that the following relations holds.

$$\mu(F_m^j(f)) + z(F_m^j(f))' = \upsilon(F_{m-1}^j(f)),\ j = 1, 2, \qquad (4.5)$$

where

(i). $\mu = \delta,\ \upsilon = \delta+1,\ for\ \ j = 1,$

and

(ii). $\mu = \delta+m-1,\ \upsilon = \delta+m,\ for\ \ j = 2.$

Now from (4.5), we can write, for $m = 1,$

$\mu F_1^j(f) + z(F_1^j(f))' = \upsilon f(z).$

Applying operator $\Im_\lambda^\beta$ 0n (4.6) and with using $p_1(z) = \dfrac{z(\Im_\lambda^\beta F_1^j(f))'}{\Im_\lambda^\beta F_1^j(f)}$, we have

$$p_1(z) + \frac{zp_1'(z)}{p_1(z)+\mu} = \frac{z(\Im_\lambda^\beta f(z))'}{\Im_\lambda^\beta f(z)}. \qquad (4.7)$$

Since $f \in \tilde{S}_\lambda(\alpha,\beta)$, we use Lemma 2.1 to have $p_1(z) \prec \tilde{p}_\alpha(z)$ in E and it implies $F_1^j(f) \in \tilde{S}_\lambda(\alpha,\beta)$. Proceeding on similar lines and using (4.5) with mathematical induction, we prove that $F_m^j(f) \in \tilde{S}_\lambda(\alpha,\beta)$ in E. □

Corollary 4.1. Let $F_{m-1}^j(f) \in \tilde{S}_\lambda(\alpha,\beta)$. Then, with $\gamma = \dfrac{9(1+\alpha)}{2(3+\alpha)^2}$,

$$\frac{z(\Im_\lambda^\beta F_m^j(f))'}{\Im_\lambda^\beta F_m^j(f)} \prec q(z) = \left\{\frac{z^{1+\mu}(1-z)^{-2(1-\gamma)}}{\int_0^z t^\mu(1-t)^{-2(1-\gamma)}dt} - 1\right\},$$

where $q(z)$ is the best dominant. The proof is immediate, when we use lemma 2.1 and subordination, see [7].

Corollary 4.1. Let $f \in \tilde{S}_\lambda(\alpha,\beta)$. Then

$$\frac{z(\Im_\lambda^\beta F_m^j(f))'}{\Im_\lambda^\beta F_m^j(f)} \prec q_m(z) = \left\{\frac{z^{1+\mu}(1-z)^{-2(1-\gamma_m)}}{\int_0^z t^\mu(1-t)^{-2(1-\gamma_m)}dt} - 1\right\},$$

where $q_m(z)$ is the best dominant.

Proof. Since $f \in \tilde{S}_\lambda(\alpha,\beta)$, it follows that

$$\frac{z(\Im_\lambda^\beta f(z))'}{\Im_\lambda^\beta f(z)} \prec \frac{1+(1-2\gamma)z}{1-z},\ \gamma = \frac{9(1+\alpha)}{2(3+\varepsilon)^\alpha}.$$

Now, as in Theorem 4.1, we have

$$\frac{z(\Im_\lambda^\beta F_1^j(f))'}{\Im_\lambda^\beta F_1^j(f)} \prec q_1(z) \prec \frac{1+(1-2\gamma)z}{1-z},$$

where $q_1(z)$ is given as

$$q_1(z) = \left\{\frac{z^{1+\mu}(1-z)^{-2(1-\gamma)}}{\int_0^z t^\mu(1-t)^{-2(1-\gamma)}dt}\right\}.$$

From (4.5), with $\dfrac{z(\Im_\lambda^\beta F_1^j(f))'}{\Im_\lambda^\beta F_1^j(f)}$, we have

$$\Re\left\{p_1(z) + \frac{zp_1'(z)}{p_1(z)+\mu}\right\} = \Re\left\{\frac{z(\Im_\lambda^\beta f(z))'}{\Im_\lambda^\beta f(z)}\right\} > \lambda.$$

Writing $p_1(z) = (1-\gamma_1)h_1(z) + \gamma_1$, we proceed to find the value of γ_1 by using previously used technique and Lemma 2.3, and have

$$\gamma_1 = \frac{2(2\mu\gamma+1)}{K_1 + \sqrt{(K_1)^2 + 8K_1}}.\ K_1 = 2\mu - 2\gamma + 1. \qquad (4.8)$$

Continuing on similar lines and using mathematical induction, we obtain the result, where

$$\gamma_m = \frac{2(2\mu\gamma+1)}{K_2 + \sqrt{(K_2)^2 + 8K_2}}.\ \ K_2 = 2\mu - 2\gamma_{m-1} + 1,$$

with $\gamma_0 = \gamma.$

Theorem 4.2. Let

$$[\Im_\lambda^\beta F(z)]^{\alpha_1} = cz^{\alpha_1-c}\int_0^z t^{(c-\alpha_1)-1}(\Im_\lambda^\beta f(t))^\delta(\Im_\lambda^\beta g(t)dt, \qquad (4.9)$$

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)

where $f, g \in \tilde{S}_\lambda(\alpha,\beta)$, $\delta+\upsilon=\alpha_1$, c,δ,α_1 and υ are positively real. Then $F \in \tilde{S}_\lambda(\alpha,\beta)$ in E.

As special case, we note that, for $\alpha_1=1,\ c=2,\ (\upsilon+\delta)=1,$ we have

$$\Im_\lambda^\beta F(z) = \frac{2}{z}\int_0^z (\Im_\lambda^\beta f(t))^\delta (\Im_\lambda^\beta f(t))^\upsilon dt.$$

Proof. We first show that $F_1 = \Im_\lambda^\beta F(z)$, defined by (4.9), is in A. Let

$f_1(z) = \Im_\lambda^\beta f(z),\ \ g_1(z) = \Im_\lambda^\beta g(z),$

and let

$$G(z) = z^{-(\delta+\upsilon)}(f_1(z))^\delta (g_1(z))^\upsilon$$
$$= 1+\alpha_2 z+\alpha_2 z^2+.....$$

and chose the branches which equals 1, when $z=0$. For

$$K(z) = z^{(c-\upsilon-\delta)-1}(f_1(z))^\delta (g_1(z))^\tau$$
$$= z^{c-1}G(z),$$

we have

$$T(z) = \frac{c}{z^c}\int_0^z K(t)dt + 1 + \frac{c}{c+1}d_1 z+.....$$

Hence $T(z)$ is well defined in E.

Now let

$$F_1(z) = [z^{\alpha_1}T(z)]^{\frac{1}{\alpha_1}} = z[T(z)]^{\frac{1}{\alpha_1}},$$

where we choose the branch of $[T(z)]^{\frac{1}{\alpha}}$ which equals 1, when $z=0$. Thus $F_1 = \Im_\lambda^\beta F \in A$ and satisfies (4.9).

Now, from 94.9) with $\dfrac{zF_1'(z)}{F_1(z)} = h(z)$, we have

$$z^{c-\alpha_1}(F_1(z))^{\alpha_1}\{(c-\alpha_1)+\alpha_1 h(z)] = c\left\{z^{(c-\delta-\upsilon)-1}(f_1(z))^\delta (g_1(z))^\upsilon\right\}. \tag{4.10}$$

We write

$$\frac{zf_1'(z)}{f_1(z)} = h_1(z),\ \ \frac{zg_1(z)}{g_1(z)} = h_2(z),$$

where $h_i(z) \prec \tilde{p}_\alpha(z),\ i=1,2.$

Differentiating (4.10) logarithmically and with some computation, we have

$$\alpha_1\left[h(z)+\frac{zh'(z)}{(c-\alpha_1+\alpha_1 h(z)}\right]$$
$$= \alpha_1(\delta+\upsilon)+\delta h_1(z)+\upsilon h_2(z).$$

That is,

$$h(z)+\frac{zh'(z)}{(c-\alpha_1+\alpha_1 h(z)}$$
$$= (1-\frac{\delta+\upsilon}{\alpha_1})\frac{\delta}{\alpha_1}h_1(z)+\frac{\upsilon}{\alpha_1}h_2(z)$$
$$= \frac{\delta}{\alpha_1}h_1(z)+\frac{\upsilon}{\alpha_1}h_2(z) = H(z).$$

The class $P(\tilde{p}_\alpha)$ is a convex set, so $H \in P(\tilde{p}_\alpha)$. This implies

$$\left[h(z)+\frac{zh'(z)}{(c-\alpha_1+\alpha_1 h(z)}\right] \prec \tilde{p}_\alpha(z) \text{ in } E.$$

Uisng Lemma 2.1, we now obtain $h(z) \prec \tilde{p}_\alpha(z)$ and this proves $F_1 \in \tilde{S}_\lambda(\alpha,\beta)$ for $z \in E$. This completes the proof. □

We now consider a partial converse of Theorem 4.2 as follows.

Theorem4.3. Let $F \in \tilde{S}_\lambda(\alpha,\beta),\ c \geq \alpha_1 > 0;\ \delta < \alpha_1(1-\gamma)$ and let f be defined as

$$(F_1(z))^{\alpha_1} = cz^{(\alpha_1}-c)\int_0^z t^{(c-\delta)-1}(f_1(t))^\delta dt, \tag{4.11}$$

where

$$F_1(z) = \Im_\lambda^\beta F(z),\quad f_1(z) = \Im_\lambda^\beta f(z).$$

Then $f_1 \in S^*(\sigma)$ for $|z| < r_0$, where r_0 is given by (2.3) in Lemma 2.5 with

$$\xi = \frac{c-\alpha_1(1-\gamma)}{\alpha_1(1-\gamma)}(\neq -1),$$

$$s = \frac{1}{\alpha_1(1-\gamma)} > 0$$

and $\sigma = \left(1-\dfrac{\delta}{\alpha_1(1-\gamma}\right).$

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)

Proof. Let

$$\frac{zF_1'(z)}{F_1(z)} = H(z) = (1-\gamma)h(z)+\gamma, \quad (4.12)$$

where $H \prec \tilde{p}_\alpha$ in E. This implies

$\Re H(z) > \gamma$ and therefore $\Re h(z) > 0$ in E. Differentiating (4.11) and uisng 94.12), we have

$$z^{(c-\alpha_1-1)}(F_1(z))^{\alpha_1}[(c-\alpha_1)+\alpha_1 H(z)]$$
$$= c\left[z^{(c-\delta-1)}(f_1(z))^{\delta}\right],$$

and logarithmi differentiation, together with (4.12) yields the following

$$\alpha_1 H(z) + \frac{\alpha_1 zH'(z)}{(c-\alpha_1)+\alpha_1 H(z)}$$
$$= (\alpha_1-\delta)+\delta\frac{zf_1'(z)}{f_1(z)}.$$

That is,

$$\alpha_1[H(z)+\frac{\alpha zH'(z)}{(c-\alpha_1)+\alpha_1 H(z)}]$$
$$= \delta[\frac{zf_1'(z)}{f_1(z)} - (1-\frac{\alpha_1}{\delta})]. \quad (4.13)$$

Using (4.12) in (4.13) and with some computation, we have

$$h(z) + \frac{zh'(z)}{(c-\alpha_1(1-\gamma))+\alpha_1(1-\gamma)h(z)}$$
$$= \left[(1-\frac{\delta}{\alpha_1(1-\gamma)})+\frac{\delta}{\alpha_1(1-\gamma)}\frac{zf_1'(z)}{f_1(z)}\right]$$
$$= \left[(1-\sigma)\frac{zf_1'(z)}{f_1(z)}\right].$$

Therefore

$$\Re\left[(1-\sigma)\frac{zf_1'(z)}{f_1(z)}\right] =$$
$$\Re\left[h(z)+\frac{\frac{1}{\alpha_1(1-\gamma)}zh'(z)}{h(z)+\frac{c-\alpha_1(1-\gamma)}{\alpha_1(1-\gamma)}}\right] > 0,$$

for $|z| < r_0$, where r_0 is given by (2.3), with

$$s = \frac{1}{\alpha_1(1-\gamma)} > 0, \; \xi = \frac{c-\alpha_1(1-\gamma)}{\alpha_1(1-\gamma)} \; (\neq -1).$$

This shows $f_1 = \Im_\lambda^\beta f \in S^*(\sigma)$ for $|z| < r_0$, where $\sigma = \left(1-\frac{\delta}{\alpha_1(1-\gamma)}\right)$. □

By assigning different permissible values to the parameters, we obtain several known and new results from Theorem 4.3. For example,, with $\gamma = \frac{1}{2}$ $(i.e.,\ \alpha = 0,),\ \alpha_1 = c = 1,\ \delta = \frac{1}{2}$, we get $f_1 \in S^*$ for $|z| < \frac{1}{2}$.

Acknowledgement: Authors are thankful to anonymous referees and editor for their useful comments and suggestions. Authors also would like to thank Dr. S. M. Junaid Zaidi (H.I., S.I.), Rector, COMSATS Institute of Information Technology, Pakistan for providing excellent academic and research enviroment. This research is carried out under HEC project Grant: NRPU-N0. 20-1966/R&D/11-2253.

References

1. S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math Soc . **135**(1969), 429-446.
2. B. C. Carlson, D.B. Shaffer, starlike and prestarlike hypergeometric functions, SIAM J. Math Anal. **15(4**),(1984), 737-745
3. P. Eenigenburg, S.S Miller, P.T. Mocanu, M.O. Reade, On a Briot-Bouquet differential subordination, Rev. Roumaine Math. Pures Appl. , **29, (1984),** 567-573.
4. I. B. Jung, Y. C. Kim, H. M. Srivastava, The Hardy space of analytic functions associated with certain one parameter families of integral operations, J. Math. Anal. Appl. **169**, (1993), 138-147.
5. Yi Ling, S. Ding, A class of analytic functions defined fractional derivation, J. Math. Anal. Appl. 186, (1994) ,504513.
6. 6. S. S. Miller, Differential inequalities and Caratheodory functions, Bull. Amer. Math. Soc. **81**(1975)79-81.
7. S. S. Miller, P. T. Mocanu, Univalent solution of Briot-Bouquet differential equations, Lecture Notes iMathematics, Springer Berlin, **1013**(1983)292-310.
8. X. Neng, Y. Dinggong, A class of analytic functions defined by the Carlson-Shaffer operator, Bull. Instit. Math. Acad . Sinica (New series) **2(**2007) 91-102.
9. S. Owa, H. M. Srivastava, . Univalent and starlike generalized hypergeometric functions, Canad. J. Math. **39(5)(**1987)1057-1077.
10. 10. S. Ponnusamy, V. Singh, Convolution properties of some classes of analytic functions, J. Math. Sci. **89(**1998)1008-1020.
11. S. Ruscheweyh, Convolution in Geometricc Function Theory, Sem. Math. Sup. Vol. 83, Press Univ. de Montreal, 1982.
12. S. Ruscheweyh, T. Shiel- Small, Hadamard products of Schlicht functions and the Polya- Schoenberg Conjecture, Comment. Math. Helv. **48**(1973)119-135.
13. S. Ruscheweyh, V. Singh, On certain external problems for functions with positive real parts, Proc. Amer. Math. Soc. 61(1976) 329-334.
14. S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. **49**(1975)109-115.
15. J. Sokol, A certain class of starlike functions, Comp. Math. Appl. **62**(2011) 611-619.
16. H. M. Srivastava, S. Owa, Univalent Functions, Fractional Calculus and Their Applications, Halsted Press, John Wiley and Sons, New York, Toronto, 1989.

(DOI: dx.doi.org/14.9831/1444-8939.2016/4-1/MRR.06)